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# A Theory of Reverse Retirement

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# A Theory of Reverse Retirement[*][†]

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**Abstract**

The retirement system is usually regarded as giving a fair reward for a long working career. However, only workers who have a sufficiently long life benefit from that reward, but not workers who die prematurely. In order to reexamine the fairness of retirement systems under unequal lifetime, this paper compares standard retirement (i.e. individuals work before being retired) with - hypothetical - reverse retirement (i.e. individuals are retired before working). We show that, under standard assumptions, an economy with reverse retirement, once in place, converges towards a unique stationary equilibrium. At the normative level, we show that, when labor productivity declines with age, reverse retirement cannot be optimal under the utilitarian criterion (unlike standard retirement), whereas reverse retirement can be optimal under the *ex post* egalitarian criterion (giving priority to the worst-off in realized terms). Finally, we show that there exists a set of policy instruments that allow a government to organize a successful transition from standard to reverse retirement.

*Keywords:* mortality, fairness, retirement, life cycle.

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# 1 Introduction

Historical roots of retirement systems are old. These date back, in England, to the Poor Laws (1599), which included dispositions for elderly individuals unable to work. In France, an *édit royal* of 1604 required the exploitants of a mine to dedicate 1/30th of their output to minors in need. However, those early retirement systems differ from modern systems, on the grounds that these were far from universal: Poor Laws were implemented at the parish level, whereas the French *édit royal* concerned only the mining industry.[1] Universal pension systems are more recent: Bismarck's old-age insurance in Germany dates back to 1889, while Beveridge's pension system in the U.K. dates back to 1942.

Retirement systems were introduced not only because of an insurance motive (protecting individuals against the risk of being poor at the old age), but, also, for redistributive reasons. Retirement systems allow, in theory, for both vertical redistribution (from - potentially richer - active young individuals towards inactive old individuals) and horizontal redistribution (from richer to poorer retirees thanks to non-proportional replacement rates). Distributive aspects are key elements in the design of a fair retirement system (see Schokkaert and Van Parijs 2003, Cremer and Pestieau 2011, Schokkaert et al 2017).

Studying the fairness of retirement systems raises additional difficulties when individuals differ on longevity. In particular, an important source of injustice lies in the fact that some proportion of the workforce dies *before* reaching the retirement age. For instance, in France, about 10 % of men and 4 % of women die before reaching the age of 60.[2] Those people obviously do not enjoy retirement. Thus, although the retirement system is usually presented as giving a fair reward for a long working career, the fairness of that system can be questioned, on the grounds that only workers who have a sufficiently long life can benefit from that reward, whereas those who die prematurely are not rewarded.

The goal of this paper is to reexamine the fairness of retirement systems in an economy with unequal lifetimes. In particular, we propose to study the capacity of retirement systems - differing in terms of the age of entry and exit of labor - to compensate individuals who turn out to die prematurely.

The reason why we focus on the compensation of the prematurely dead is the following. Actually, in advanced societies (with high productivity), the worst-off individual is, in general, the short-lived. Moreover, longevity inequalities are largely due to circumstances, i.e. factors on which individuals have little control (such as genetic background or environmental quality).[3] Given that well-being inequalities related to unequal longevity are due to circumstances, it follows that the Principle of Compensation applies (Fleurbaey and Maniquet 2004 and Fleurbaey 2008). That principle, which states that inequalities in well-being

[1] Another *édit royal* published in 1673 created a pension for officers of the *Marine Royale*, while pensions were introduced for soldiers and civil servants in, respectively, 1831 and 1853 (see Lavigne 2013).

[2] This statement is based on the 2014 lifetable. Sources: The Human Mortality Database.

[3] On the impact of genetic background on longevity inequalities, see Christensen et al (2006).

due to circumstances should be abolished by governments, provides an ethical support for compensating individuals for unequal lifetimes.

While there is an ethical support for compensating the short-lived, it is not trivial to see how governments could proceed to achieve such a compensation. In particular, how could one design retirement systems favoring the compensation of the short-lived? In a recent paper, Fleurbaey et al (2016) examined how varying the age at retirement could achieve such a compensation. For that purpose, they characterized the optimal retirement age while adopting an *ex post* egalitarian social welfare criterion, which gives absolute priority to the worst-off in realized terms (who is, in general, the short-lived). Fleurbaey et al (2016) showed that the compensation of the unlucky short-lived pushes towards *postponing* retirement in comparison with the utilitarian social optimum. The underlying intuition is that postponing retirement allows to transfer more resources towards the young age, and, hence, to increase the well-being of all young individuals (including those who will turn out to die prematurely).

Although Fleurbaey et al (2016) casts some light on how taking care about the compensation of the short-lived can affect the optimal age at retirement, it remained based on the standard view on retirement. Actually, Fleurbaey et al (2016) assumed the usual life cycle, where individuals work at the young age, and become retiree as they reach some (older) age. But is this standard retirement system the only possible one? Can we think about alternative retirement systems that would be more fair with respect to the unlucky short-lived?

In order to reexamine the fairness of retirement systems under unequal lifetimes, this paper proposes to go beyond the standard representation of retirement systems, in which individuals are first workers at the young age, and, then, if they survive to sufficiently high ages, retirees. We propose to consider also what we call a "reverse" retirement system, where individuals would - unlike in existing societies - be first retirees at the young age, and, then, workers at the old age. This paper examines the conditions under which such a - purely hypothetical - reverse retirement system dominates the standard retirement system.

At this early stage of our explorations, it should be stressed that reverse retirement does not exist in actual economies, and is thus a pure theoretical abstraction. Reverse retirement is a kind of "utopia", in the same way as standard retirement was regarded as an utopia during the longest part of History.[4] Note, however, that, in the common language, the term "reverse retirement" refers to the behavior of a minority of retirees who go back to work. In some sense, the reverse retirement system that we consider is a generalization of this behavior to the entire society.[5] We propose to compare existing standard retirement systems with the - purely hypothetical - reverse retirement system.

[4] Although reverse retirement does not exist in actual economies, it is sometimes mentioned as a fanciful utopia, for instance by humorists. An example is the reform of reversing retirement introduced in the hypothetical country "Groland" of the French humorists of the Canal + TV channel. This TV show presents, as a parody, elderly workers in bad health with low productivity, who are serving young healthy people enjoying leisure. This parody illustrates how perceptions about age can interplay with beliefs about desirable social architecture.

[5] Here again, there is an obvious parallel with standard retirement, which was introduced initially for some jobs, before being generalized to the whole society in the 20th century.

For that purpose, we develop a 4-period overlapping generations model (OLG) with unequal lifetime. We assume that: (1) production involves physical capital as well as young and/or old labor; (2) there is a perfect substitutability between young and old labor (but with age-dependent labor productivity); (3) lifetime well-being is the sum, across periods, of temporal well-being; (4) temporal well-being is increasing in consumption and in leisure time; (5) older workers face a higher marginal disutility of labor than younger workers.

We first study, under assumptions (1)-(5), the temporary equilibrium, as well as the long-run dynamics of the economy, under either standard retirement or reverse retirement. Those two kinds of economy differ from a qualitative perspective: under standard retirement, young individuals work and save for their old days (during which they will be retired), whereas, under reverse retirement, young individuals do not work, borrow resources, and pay these back at the old age (during which they work). In a second stage, we examine the social desirability of standard and reverse retirement, under two social welfare criteria: utilitarianism and *ex post* egalitarianism.

Our results are threefold. First, at a positive level, we show that, under standard assumptions on technology and preferences, an economy with reverse retirement, once in place, *converges* towards a unique stationary equilibrium. Thus an economy with reverse retirement is clearly sustainable in the long-run. Second, at the normative level, we show that, when labor productivity declines with age, reverse retirement can never be optimal under the utilitarian criterion, but can, under some conditions, be optimal under the *ex post* egalitarian criterion. From the *ex post* egalitarian perspective, standard retirement dominates reverse retirement in less developed economies (characterized by harsh working conditions, a steep age-productivity profile and a lower probability of survival to age 50), but reverse retirement dominates in advanced economies (characterized by less harsh working conditions, a flatter age-productivity profile, and a higher probability of survival to age 50). Third, we show that, although the *transition* from standard to reverse retirement leads to the collapse of the economy in the laissez-faire, there exists a set of policy instruments that allow governments to organize a successful transition from standard to reverse retirement.

This paper is related to several branches of the literature. First, it is related to the literature on retirement and distribution (Schokkaert and Van Parijs 2003, Cremer and Pestieau 2011, Schokkaert et al 2017), which focuses on standard retirement, unlike this paper, which also examines reverse retirement. Second, this paper is also related to the literature on fairness and compensation under unequal lifetime (see Fleurbaey and Ponthiere 2013, Fleurbaey et al 2014, Fleurbaey et al 2016). Fleurbaey et al (2016) examined, within a standard retirement system, the choice of the optimal retirement age under the *ex post* egalitarian social criterion. Here we propose to compare standard retirement with reverse retirement from the perspective of the interests of the short-lived. Finally, this paper complements also Leroux and Ponthiere (2018), which studied, in a static model, the design of optimal working time regulations while focusing on the *intensive* margins of labor (i.e. number of hours worked per week). The present paper, on the contrary, studies the design of optimal *extensive* margins of labor (i.e. ages of entry and exit from labor) in a dynamic framework.

The paper is organized as follows. Section 2 presents the model. Section 3 characterizes the temporary equilibrium, and examines conditions under which it involves standard or reverse retirement. That section examines also the long-run dynamics of the economy, under standard retirement or reverse retirement. Section 4 characterizes the long-run utilitarian optimum and the long-run *ex post* egalitarian optimum, and examines the conditions under which these involve either standard or reverse retirement. The decentralization of those social optima is studied in Section 5. Section 6 examines criticisms against reverse retirement, both at positive and normative levels. Section 7 concludes.

# 2 The model

Let us consider a 4-period OLG economy. Fertility is at the replacement level (one child per individual), and each cohort has a size $N > 0$.[6] Time is discrete and goes from 0 to $+\infty$. Each time period has a unitary length.[7]

During period 1 (childhood), no decision is made. During period 2 (young adulthood), individuals plan their entire life. In period 2, individuals have one child, consume and save some resources for old days. Old adulthood (period 3) is reached with a probability $\pi$, with $0 < \pi < 1$. In period 3, individuals consume and save for the very old age. Conditionally on survival to the old age, period 4 (the very old age) is reached with probability $p$, with $0 < p < 1$. During the very old age, individuals are dependent and can only consume.

Labor does not take place in period 1 (childhood) and period 4 (very old age). Labor can only take place in intermediate life-periods, i.e. periods 2 and 3. Under standard retirement (no old-age labor), the age of entry in the labor market is age 1, and the age of exit of the labor market (retirement age) is $1+\ell_t$, where $0 \leq \ell_t \leq 1$. Under reverse retirement (no young-age labor), the age of entry in the labor market is age 2, and the age of exit of the labor market is $2 + \tilde{\ell}_{t+1}$, where $0 \leq \tilde{\ell}_{t+1} \leq 1$.[8]

**Production** Production takes place by using physical capital and labor, according to the following production function:

$$Y_t = F(K_t, L_t) \tag{1}$$

where $F(\cdot)$ is increasing and concave in its arguments, capital $K_t$ and labor $L_t$, and exhibits constant returns to scale.

[6] We do not consider here the issue of optimal fertility. See Pestieau and Ponthiere (2017) on optimal fertility under age-dependent labor productivity.

[7] That 4-period OLG model is a reduced form of the life cycle. One can, for instance, interpret period 1 as going from age 0 to age 25, period 2 (young adulthood) as going from age 25 to age 50, period 3 (old age) as going from age 50 to age 75, and period 4 (very old age) as going from age 75 to age 100.

[8] We do not exclude, at this early stage, the possibility of individuals working in *both* periods 2 and 3. In that case, the age of entry on the labor market is $1 + (1 - \ell_t) = 2 - \ell_t$, whereas the age of exit from the labor market is $2 + \tilde{\ell}_{t+1}$.

Capital fully depreciates after one period of use.

Moreover, we assume that there is perfect substitutability between labor at the young age and labor at the old age:

$$L_t = aN\ell_t + b\pi N\tilde{\ell}_t \tag{2}$$

where $a > 0$ and $b > 0$ account for, respectively, the productivity of labor at the young age and at the old age.

Empirical studies provide mixed results regarding the link between age and labor productivity. Haegeland et Klette (1999) show that older workers are more productive than younger workers, whereas Crepon et al (2003) show that productivity exhibits an inverted U shaped curve with the age. Aubert and Crépon (2007) and Gobel et Zwick (2009) find that productivity grows with age until age 45, and then stabilizes.

Given that the empirical literature on the link between age and productivity provides mixed results, we will, in this paper, consider all possible cases, where either young workers are at least as productive as old workers, that is, $a \geq b$, or where old workers are more productive than young workers, i.e. $a < b$.

**Preferences** In order to study the social desirability of standard or reverse retirement, it is necessary to specify individual preferences while allowing for the disutility of labor to vary with age, since a higher disutility of old-age labor is often regarded as a major justification of standard retirement.[9]

In young adulthood, well-being $U_t^y$ is equal to:

$$U_t^y = u(c_t) - v\ell_t \tag{3}$$

where $c_t$ is consumption in young adulthood, $u(\cdot)$ is increasing and concave, and $v > 0$ reflects the (marginal) disutility of working. As usual, we assume that there exists a level of consumption $\bar{c} > 0$ such that $u(\bar{c}) = 0$.

A the old age, individual well-being $U_t^o$ is equal to:

$$U_t^o = u(d_t) - \tilde{v}\tilde{\ell}_t \tag{4}$$

where $d_t$ is consumption at old age, while $\tilde{v} > 0$ reflects the (marginal) disutility of old-age labor.

Without loss of generality, we assume that the marginal disutility of working at the old age is larger than the marginal disutility of working at the young age:

$$\tilde{v} > v \tag{5}$$

At the very old age (period 4), individuals just care about their consumption:

$$U_t^{vo} = u(e_t) \tag{6}$$

where $e_t$ is consumption at the very old age.

[9] For the sake of analytical simplicity, the consumption component of well-being is assumed to be age-invariant. Relaxing that assumption would complicate the analysis without bringing new insights for the issue at stake.

# 3 The laissez-faire

Let us first study a perfectly competitive economy, where production factors are paid at their marginal productivity:

$$\begin{aligned} w_t &= aF_L\left(K_t, aN\ell_t + b\pi N\tilde{\ell}_t\right) & (7)\\ \tilde{w}_t &= bF_L\left(K_t, aN\ell_t + b\pi N\tilde{\ell}_t\right) & (8)\\ R_t &= F_K\left(K_t, aN\ell_t + b\pi N\tilde{\ell}_t\right) & (9) \end{aligned}$$

where $w_t$ is the wage rate for the young worker, $\tilde{w}_t$ is the wage rate for the old worker, and $R_t$ equals 1 *plus* the interest rate.

We assume that there exists a perfect annuity market, which yields an actuarially fair return. The return on savings for young adults is:

$$\hat{R}_t = \frac{R_t}{\pi} \tag{10}$$

where $\hat{R}_t$ denotes the gross interest factor, inclusive of the survival premium.

The return on savings for old adults is equal to:

$$\check{R}_t = \frac{R_t}{p} \tag{11}$$

where $\check{R}_t$ denotes the gross interest factor, inclusive of the survival premium.

## 3.1 The temporary equilibrium

The individual who is a young adult at time $t$ chooses savings $s_t$ and $z_{t+1}$, as well as young-age working period $\ell_t$ and old-age working period $\tilde{\ell}_{t+1}$, so as to maximize his expected lifetime welfare, while taking market prices as given. The young adult at time $t$ forms anticipations about future interest factors $R_{t+1}^{E_t}$ and $R_{t+2}^{E_t}$, and about the future wage at the old age, i.e. $\tilde{w}_{t+1}^{E_t}$.

The problem of the young adult at time $t$ is:[10]

$$\max_{s_t, z_t, \ell_t, \tilde{\ell}_{t+1}} \left[\begin{array}{l} u\left(w_t\ell_t - s_t\right) - v\ell_t + \pi\left[u\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t}s_t}{\pi} - z_{t+1}\right) - \tilde{\ell}_{t+1}\tilde{v}\right] \\ +\pi p u\left(\frac{R_{t+2}^{E_t}z_{t+1}}{p}\right) \end{array}\right]$$

$$\begin{aligned} \text{s.t. } \ell &\geq 0 \text{ and } 1-\ell \geq 0 \\ \text{s.t. } \tilde{\ell} &\geq 0 \text{ and } 1-\tilde{\ell} \geq 0 \end{aligned}$$

That problem is solved in the Appendix. Proposition 1 characterizes the temporary equilibrium that prevails at time $t$ at the laissez-faire.

[10] Note that we do not impose a non-negativity constraint on savings, since in case of reverse retirement young-age consumption is satisfied by borrowing resources, which are paid back at the old age (see below).

**Proposition 1 (laissez-faire temporary equilibrium)** *Consider the temporary equilibrium at time t given anticipations* $\left\{R_t^{E_{t-1}}, R_{t+1}^{E_{t-1}}, R_{t+1}^{E_t}, R_{t+2}^{E_t}, \tilde{w}_t^{E_{t-1}}, \tilde{w}_{t+1}^{E_t}\right\}$.

- *If* $\frac{v}{\tilde{v}} < \frac{R_t^{E_{t-1}} w_{t-1}}{\tilde{w}_t^{E_{t-1}}}$ *and* $\frac{v}{\tilde{v}} < \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}}$, *standard retirement prevails* ($\tilde{\ell}_t = \tilde{\ell}_{t+1} = 0$), *and* $\{\ell_{t-1}, \ell_t, K_t, s_{t-1}, s_t, z_t, z_{t+1}, w_t, R_t\}$ *satisfy:*

$$\begin{aligned}
u'\left(w_t \ell_t - s_t\right) &= R_{t+1}^{E_t} u'\left(\frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) = R_{t+1}^{E_t} R_{t+2}^{E_t} u'\left(\frac{R_{t+1}^{E_t} z_{t+1}}{p}\right) \\
u'\left(\frac{R_t^{E_{t-1}} s_{t-1}}{\pi} - z_t\right) &= R_{t+1}^{E_{t-1}} u'\left(\frac{R_t^{E_{t-1}} z_t}{p}\right) \\
w_{t-1} u'\left(w_{t-1}\ell_{t-1} - s_{t-1}\right) &\geq v \text{ and } w_t u'\left(w_t \ell_t - s_t\right) \geq v \\
K_t &= N s_{t-1} + \pi N z_{t-1} \\
w_t &= a F_L\left(K_t, aN\ell_t\right) \text{ and } R_t = F_K\left(K_t, aN\ell_t\right)
\end{aligned}$$

- *If* $\frac{v}{\tilde{v}} > \frac{R_t^{E_{t-1}} w_{t-1}}{\tilde{w}_t^{E_{t-1}}}$ *and* $\frac{v}{\tilde{v}} > \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}}$, *reverse retirement prevails* ($\ell_{t-1} = \ell_t = 0$), *and* $\left\{\tilde{\ell}_t, \tilde{\ell}_{t+1}, K_t, s_{t-1}, s_t, z_t, z_{t+1}, w_t, R_t\right\}$ *satisfy:*

$$\begin{aligned}
u'\left(-s_t\right) &= R_{t+1}^{E_t} u'\left(\begin{array}{l} \tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi} \\ -z_{t+1} \end{array}\right) = R_{t+1}^{E_t} R_{t+2}^{E_t} u'\left(\frac{R_{t+2}^{E_t} z_{t+1}}{p}\right) \\
u'\left(\begin{array}{l} \tilde{w}_t^{E_{t-1}}\tilde{\ell}_t + \frac{R_t^{E_{t-1}} s_{t-1}}{\pi} \\ -z_t \end{array}\right) &= R_{t+1}^{E_{t-1}} u'\left(\frac{R_t^{E_{t-1}} z_t}{p}\right) \\
\tilde{w}_{t+1}^{E_t} u'\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi}\right) &\geq \tilde{v} \text{ and } \tilde{w}_t^{E_{t-1}} u'\left(\tilde{w}_t^{E_{t-1}}\tilde{\ell}_t + \frac{R_t^{E_{t-1}} s_t}{\pi}\right) \geq \tilde{v} \\
K_t &= N s_{t-1} + \pi N z_{t-1} \\
\tilde{w}_t &= b F_L\left(K_t, b\pi N\tilde{\ell}_t\right) \text{ and } R_t = F_K\left(K_t, b\pi N\tilde{\ell}_t\right)
\end{aligned}$$

**Proof.** See the Appendix. ■

Proposition 1 states that whether the laissez-faire temporary equilibrium involves standard retirement ($\tilde{\ell}_t = \tilde{\ell}_{t+1} = 0$) or reverse retirement ($\ell_{t-1} = \ell_t = 0$) depends on several factors.[11]

First, this depends on how large the disutility of old-age labor $\tilde{v}$ is with respect to the disutility of young-age labor $v$. When the former is much larger than the latter, then standard retirement prevails at the laissez-faire.

Whether standard or reverse retirement prevails depends also on wages, which, in a competitive economy, are equal to the marginal productivity of labor. If young workers are more productive than old workers (i.e. $a > b$),

[11] Note that Proposition 1 does not consider the cases where successive cohorts disagree about standard/reverse retirement. This point is discussed in Section 3.2.

this pushes, *ceteris paribus*, towards $w_t > \tilde{w}_{t+1}^{E_t}$, and, hence, towards standard retirement. On the contrary, if young workers are less productive than old ones (i.e. $a < b$), this pushes towards $w_t < \tilde{w}_{t+1}^{E_t}$, and, hence, to reverse retirement.

One should also notice the role of the interest factor, $R_{t+1}^{E_t}$. Clearly, if there is underaccumulation of capital, then $R_{t+1}^{E_t} > 1$, which favors, *ceteris paribus*, standard retirement. On the contrary, when there is overaccumulation of capital, $R_{t+1}^{E_t} < 1$, this favors reverse retirement.

Proposition 1 can be used to explain or "rationalize" the real-world economy, where standard retirement prevails. Clearly, we are in underaccumulation of capital, leading to $R_{t+1}^{E_t} > 1$, so that, given $\tilde{v} > v$, we have $\frac{v}{\tilde{v}} < R_{t+1}^{E_t}$. Hence, provided the wage profile is not too increasing with age, the condition $\frac{v}{\tilde{v}} < \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}}$ is satisfied, leading to standard retirement. The intuition behind that result is the following. When there is underaccumulation of capital, the interest rate is high, and so, given that working at the young age brings less disutility (in comparison to working at the old age), and does not bring a too low income in comparison to working at the old age, it is optimal to work and save when being young, and, then, to retire at the old age. Such a temporary equilibrium coincides with what is observed in real-world economies, where young adults work and save, whereas old adults are retired.

Although that case matches with real-world economies, one can also consider, in theory, an economy with reverse retirement. Under overaccumulation of capital ($R_{t+1}^{E_t} < 1$), it is not necessarily the case that standard retirement prevails at the temporary equilibrium: provided the disutility of old-age labor is not so high in comparison to the disutility of young-age labor, it is possible that reverse retirement prevails when the age-productivity gap is not strong. That situation is quite different from real-world economies: here individuals would not work at the young age, and would borrow resources and consume these (i.e. negative savings: $s_t < 0$). Then, at the old age, individuals would work and pay their debt back (with the associated interests).

In sum, whether standard or reverse retirement prevails at the temporary equilibrium depends on three factors. First, it depends on preferences, in particular on how large the disutility of old-age labor $\tilde{v}$ is with respect to the disutility of young-age labor $v$. Second, it depends on the age-productivity gap, that is, on whether $a \gtreqless b$, which affects the wage gap. Thirdly, it depends on the extent to which the economy is in under- or overaccumulation of capital.

## 3.2 Long-run dynamics

Having studied the temporary equilibrium, let us now examine the long-run dynamics of our economy. Given that there are possibly two kinds of retirement regimes prevailing at the temporary equilibrium - either standard or reverse retirement -, one cannot exclude, in theory, that at some point there could be a *transition* from one retirement regime to another. Such a shift could arise, for instance, if we have:

$$\frac{v}{\tilde{v}} < \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E}} \text{ and } \frac{v}{\tilde{v}} > \frac{R_{t+2}^{E_{t+1}} w_{t+1}}{\tilde{w}_{t+2}^{E_{t+1}}}$$

leading to standard retirement chosen by young adults at time $t$, and reverse retirement chosen by young adults at time $t+1$.[12]

Retirement regime shifts would raise serious problems in the laissez-faire economy. To see this, let us consider an economy with standard retirement at time $t$, but with a transition to reverse retirement at time $t+1$. In period $t+1$, a fundamental problem would arise: there would be, during that transition period, no worker on the labor market, since the cohort of young adults at time $t$ would be retired at time $t+1$, whereas the cohort of young adults at time $t+1$ would work only at the old age, that is, in period $t+2$.[13] As a consequence, in the laissez-faire, the transition from standard retirement to reverse retirement would lead to the collapse of the economy, because of the absence of labor (and, thus, of production) during one period of time.

Thus transitions between retirement regimes raise important difficulties at the laissez-faire. This does not imply, of course, that such transitions are impossible, but simply that these transitions require some public intervention, and cannot be left to individual decisions only. We will turn back to the transition issue when considering the decentralization of the *ex post* egalitarian optimum with reverse retirement in Section 5.

In order to avoid difficulties raised by retirement regime shifts at the laissez-faire, we will simply assume that expectations about future factor prices are such that regime shifts cannot arise in the laissez-faire. For that purpose, we impose the following non-regime shift condition.

**Definition 1 (the non-regime shift condition)** *Individual expectations on future factor prices* $\left\{\tilde{w}_{t+1}^{E_t}, R_{t+1}^{E_t}\right\}$ *satisfy the conditions:*

$$\begin{aligned} \text{If, at } t &= 0,\ \frac{v}{\tilde{v}} < \frac{R_1^{E_0} w_0}{\tilde{w}_1^{E_0}}, \text{ then, for all } t > 0, \text{ we have } \frac{v}{\tilde{v}} < \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}}; \\ \text{If, at } t &= 0,\ \frac{v}{\tilde{v}} > \frac{R_1^{E_0} w_0}{\tilde{w}_1^{E_0}}, \text{ then, for all } t > 0, \text{ we have } \frac{v}{\tilde{v}} > \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}}. \end{aligned}$$

That condition on expectations about future factor prices guarantees that, once the temporary equilibrium involves a particular retirement system, this

[12] Alternatively, one could have the opposite shift, from reverse to standard retirement, if:

$$\frac{v}{\tilde{v}} > \frac{R_{t+1}^{E_t} w_t}{\tilde{w}_{t+1}^{E_t}} \text{ and } \frac{v}{\tilde{v}} < \frac{R_{t+2}^{E_{t+1}} w_{t+1}}{\tilde{w}_{t+2}^{E_{t+1}}}$$

[13] On the contrary, if there were a transition from a reverse retirement regime to a standard retirement regime, there would be a period without retirees (except the very old).

particular retirement system will, in the absence of any parameter change (or any public intervention), keep on prevailing in the future.

Under that non-regime shift condition, we can study the dynamics of the economy *conditionally on a given retirement regime*, without worrying about shifts or transitions from one regime to another.

For that purpose, let us consider the dynamics of capital accumulation, conditionally on a given retirement regime. That dynamics is given by the law:

$$K_{t+1} = Ns_t + \pi N z_t \tag{12}$$

The first term on the RHS is the saving $s_t$ from the young adults, which is positive under standard retirement, but negative under reverse retirement, in the sense that young adults are then borrowing to finance young-age consumption (since they do not work). The second term of the RHS is the savings of the old adults $z_t$, which is always positive, whatever we consider standard or reverse retirement. The underlying intuition is that individuals at the very old age cannot work any more. Hence, consumption in period 4 of life must be based on the savings of period 3, which must thus be positive.

The above expression can also be interpreted as the capital market equilibrium condition. The supply of capital comes from the saving of the young adults and/or old adults, and can be used either to finance the investment in the productive capacity, or to finance the consumption of the young under reverse retirement. In any case, in equilibrium there must be an equality of demand and supply of capital, which is given by the above equality.

The long-run dynamics of the economy can only be studied provided we make assumptions on how individuals form anticipations, and, also, provided we make additional assumptions on the production technology and on preferences. Proposition 3 assumes that individuals have perfect foresight, so that: $R^{E_t}_{t+1} = R_{t+1}$, $R^{E_t}_{t+2} = R_{t+2}$ and $\tilde{w}^{E_t}_{t+1} = \tilde{w}_{t+1}$. It assumes also that $u(c_t)$ has a logarithmic form, as well as a Cobb-Douglas technology.

**Proposition 2 (laissez-faire stationary equilibrium)** *Consider the stationary equilibrium with perfect foresight with* $\max\left\{\ell_t, \tilde{\ell}_t\right\} < 1$. *Assume the non-regime shift condition, as well as* $u(c_t) = \log(c_t) - \beta$ *and* $Y_t = AK_t^{\alpha}\left(aN\ell_t + b\pi N\tilde{\ell}_t\right)^{1-\alpha}$ *with* $0 < \alpha < \frac{1}{2}$.

- *If the laissez-faire temporary equilibrium at* $t = 0$ *involves standard retirement (i.e.* $\ell_0 > 0, \tilde{\ell}_0 = 0$*), there exist only two stationary equilibria* $K^{s*} = 0$ *and* $K^{s**} > 0$*, where* $K^{s*}$ *is unstable, while* $K^{s**}$ *is locally stable.*

- *If the laissez-faire temporary equilibrium at* $t = 0$ *involves reverse retirement (i.e.* $\ell_0 = 0, \tilde{\ell}_0 > 0$*), there exist only two stationary equilibria* $K^{r*} = 0$ *and* $K^{r**} > 0$*, where* $K^{r*}$ *is unstable, while* $K^{s**}$ *is locally stable.*

**Proof.** See the Appendix. ■

Proposition 2 states that, when the temporary equilibrium exhibits standard retirement, the economy exhibits two stationary equilibria, and the economy will, from any initial condition $K_0 > 0$, converge towards the steady-state $K^{s**} > 0$. The level of the steady-state capital stock is increasing with the probability to reach the old age $\pi$, which increases the propensity to save.

Quite interestingly, Proposition 2 states that, when reverse retirement prevails at the temporary equilibrium, the economy exhibits also two stationary equilibria, and the economy will, from any initial condition $K_0 > 0$, converge towards the stationary equilibrium $K^{r**} > 0$. That result of existence, uniqueness and stability of the stationary equilibrium with strictly positive capital stock is quite important, since one may believe, at first glance, that reverse retirement is a kind of "utopia", which would not be sustainable. Proposition 2 states, on the contrary, that an economy with reverse retirement converges, in the long-run, towards a unique stationary equilibrium. Thus, once in place, an economy with reverse retirement would not collapse into instability and chaos.

# 4 The long-run social optimum

Let us now characterize the long-run social optimum of our economy. That characterization is a preliminary stage to the design of optimal public policies. For that purpose, we will proceed in two stages. We will first consider the long-run utilitarian social optimum, where the social objective is the maximization of total welfare at the stationary equilibrium. Then, in a second stage, we will consider the *ex post* egalitarian social optimum, where the social objective is the maximization of the minimum welfare level at the stationary equilibrium.

## 4.1 The utilitarian optimum

Under the utilitarian social criterion (Bentham 1789, Mill 1863), the social objective is the maximization of the sum of individual utilities. In our context, the utilitarian social planner chooses $\left\{c, d, e, \ell, \tilde{\ell}, K\right\}$ so as to maximize the sum of individual utilities at the stationary equilibrium, subject to the resource constraint of the economy. That social planning problem can be written as:[14]

$$\begin{array}{rrl} \max\limits_{c,d,e,\ell,\tilde{\ell},K} & & N\left[u\left(c\right) - v\ell + \pi\left[u(d) - \tilde{v}\tilde{\ell}\right] + \pi p u(e)\right] \\ \text{s.t.} & & F\left(K, aN\ell + b\pi N\tilde{\ell}\right) = Nc + \pi Nd + \pi pNe + K \\ \text{s.t.} & \ell \geq & 0 \text{ and } 1 - \ell \geq 0 \\ \text{s.t.} & \tilde{\ell} \geq & 0 \text{ and } 1 - \tilde{\ell} \geq 0 \end{array}$$

That problem is solved in the Appendix. Our results are summarized in Proposition 3.[15]

[14] We abstract from time indexes, since we consider a stationary economy.

[15] We omit here the case where $\frac{v}{a} = \frac{\tilde{v}}{b}$, in which case the optimal retirement system is indeterminate.

**Proposition 3** *Consider the long-run utilitarian social optimum* $\left\{c^u, d^u, e^u, \ell^u, \tilde{\ell}^u, K^u\right\}$.

- *If young workers are weakly more productive than old workers (i.e.* $a \geq b$*), then standard retirement prevails (i.e.* $\tilde{\ell}^u = 0$*), and we have:*

$$\begin{aligned} u'(c^u) &= u'(d^u) = u'(e^u) \\ u'(c^u) F_L\left(K^u, aN\ell^u\right) a &\geq v \text{ and } F_K\left(K^u, aN\ell^u\right) = 1 \end{aligned}$$

- *If old workers are more productive than young workers (i.e.* $a < b$*), then:*

  – *If* $\frac{v}{a} < \frac{\tilde{v}}{b}$*, standard retirement prevails (i.e.* $\tilde{\ell}^u = 0$*), and we have:*

$$\begin{aligned} u'(c^u) &= u'(d^u) = u'(e^u) \\ u'(c^u) F_L\left(K^u, aN\ell^u\right) a &\geq v \text{ and } F_K\left(K^u, aN\ell^u\right) = 1 \end{aligned}$$

  – *If* $\frac{v}{a} > \frac{\tilde{v}}{b}$*, reverse retirement prevails (i.e.* $\ell^u = 0$*), and we have:*

$$\begin{aligned} u'(c^u) &= u'(d^u) = u'(e^u) \\ u'(c^u) F_L\left(K^u, b\pi N\tilde{\ell}^u\right) b &\geq \tilde{v} \text{ and } F_K\left(K^u, \pi N\tilde{\ell}^u\right) = 1 \end{aligned}$$

**Proof.** See the Appendix. ■

Proposition 3 states that, when young workers are weakly more productive than old workers, the long-run utilitarian optimum does not involve reverse retirement, but necessarily involves standard retirement.

In order to understand the intuition behind that result, it is useful to proceed by *reductio ad absurdum.* Let us consider an economy with reverse retirement. At zero labor for the young, transferring one unit of labor from an old to a young brings, at the margin, more output (since the young is more productive than the old) and implies also a lower disutility of labor (since the marginal disutility of labor is larger for the old than for the young). Hence, if transferring one unit of labor from an old to a young increases output and reduces labor disutility, the initial situation cannot be optimal from a utilitarian perspective. Thus it has to be that the young should work at least some positive time-period (i.e. $\ell > 0$) at the utilitarian optimum.

However, when old workers are more productive than young workers, then it is possible that reverse retirement is part of the utilitarian optimum, but only if the gap in labor productivity between the old and the young is strong enough in comparison with the gap in disutility of labor. Thus the utilitarian social criterion does not necessarily exclude reverse retirement, but reverse retirement can only be socially optimal provided there is a sufficiently strong productivity advantage for old workers in comparison to young workers, leading to $\frac{v}{a} > \frac{\tilde{v}}{b}$.

While Proposition 3 provides a general characterization of the long-run utilitarian social optimum under either standard or reverse retirement, Proposition 4 shows the explicit form of the long-run utilitarian optimum under a logarithmic utility function $u(\cdot)$ and a Cobb-Douglas production function.

**Proposition 4** *Assume* $u(c_t) = \log(c_t) - \beta$ *and* $Y_t = AK_t^{\alpha}\left(aN\ell_t + b\pi N\tilde{\ell}_t\right)^{1-\alpha}$ *with* $0 < \alpha < \frac{1}{2}$. *Assume that* $\max\left\{\ell, \tilde{\ell}\right\} < 1$. *Define* $\Xi \equiv A(1-\alpha)(A\alpha)^{\frac{\alpha}{1-\alpha}}$.

- *At the long-run utilitarian optimum with standard retirement (i.e.* $\tilde{\ell}^u = 0$*):*

$$c^u = d^u = e^u = \frac{a}{v}\Xi;\ \ell^u = \frac{1+\pi+\pi p}{v};\ K^u = aN(A\alpha)^{\frac{1}{1-\alpha}}\frac{(1+\pi+\pi p)}{v}$$

- *At the long-run utilitarian optimum with reverse retirement (i.e.* $\ell^u = 0$*):*

$$c^u = d^u = e^u = \frac{b}{\tilde{v}}\Xi;\ \tilde{\ell}^u = \frac{1+\pi+\pi p}{\pi\tilde{v}};\ K^u = \pi bN(A\alpha)^{\frac{1}{1-\alpha}}\frac{(1+\pi+\pi p)}{\pi\tilde{v}}$$

**Proof.** See the Appendix. ■

In the light of Proposition 4, it appears that the two kinds of long-run utilitarian optima present quite symmetric forms. When considering the optimal consumption profile and the optimal capital stock, these differ only up to a factor $\frac{a}{v}$ or $\frac{b}{\tilde{v}}$, which are the factors that already determined, in Proposition 3, whether the long-run utilitarian optimum involved either standard or reverse retirement.

It is not clear whether the utilitarian optimum with standard retirement involves a longer career than the one with reverse retirement. This depends on whether $v$ is inferior to $\pi\tilde{v}$ or not. Note that, under standard retirement, the optimal age at retirement $1+\ell^u$ is increasing in the survival probability to the old age $\pi$, and is also increasing in the (conditional) survival probability to the very old age $p$. Under reverse retirement, the optimal age at retirement $1+\ell^u$ is also increasing in $p$, but is decreasing in $\pi$.

## 4.2 The ex post egalitarian optimum

The utilitarian social criterion has become a kind of benchmark normative criterion in public economics. However, when considering environments where individuals have unequal lifetimes, the utilitarian criterion yields somewhat counterintuitive results, since this criterion does not do justice to the idea of compensating the unlucky short-lived.

To see this, note that the long-run utilitarian optimum involves perfect smoothing of consumption along the entire life. This perfect smoothing leads to large well-being losses in case of premature death. The $(1-\pi)N$ individuals who die before reaching the old age only consume $c^u$ during their life, whereas the rest of the population consumes either $2c^u$ (for individuals who do not reach the very old age) or even $3c^u$ (for individuals who reach the very old age). Thus consumption smoothing penalizes individuals who die before the old age.

In addition, note also that the utilitarian optimum involves also, under general conditions (i.e. $\frac{v}{a} < \frac{\tilde{v}}{b}$), standard retirement, i.e. the working time is concentrated on young adulthood. This is also a major source of deprivation for the unlucky short-lived. Those $(1-\pi)N$ individuals who die before reaching the old age produce resources, but do not enjoy retirement. This is, in addition to consumption smoothing, a second source of well-being losses for the unlucky short-lived in comparison to long-lived individuals. Obviously, this second source of deprivation does not arise if the utilitarian optimum involves reverse retirement, but even in that case short-lived individuals suffer from well-being losses due to consumption smoothing.

One could defend the utilitarian optimum by replying to those criticisms that no one can identify, *ex ante*, who will be short-lived, and that those unlucky short-lived individuals are simply the victims of this lack of knowledge. But that argument is not convincing: even if no one knows *ex ante* who will be short-lived and who will be long-lived, it is possible, by reorganizing the life cycle (in terms of consumption profiles and working periods), to minimize the well-being losses for the unlucky short-lived. As shown in Fleurbaey et al (2014, 2016), this task can be done by giving up the utilitarian criterion, and by adopting a social criterion that does more justice to the idea of compensating the short-lived.

This motivates the use of an alternative social criterion, which gives more weight to the short-lived. Actually, the *ex post* egalitarian social welfare criterion (see Fleurbaey et al 2014, 2016) gives absolute priority to the worst off individual in realized terms (rather than in expected terms). Within our model, the worst off in realized terms is, under general conditions, the short-lived (i.e. individuals who die before reaching the old age). Hence that social criterion amounts to give priority to the interests of short-lived individuals, and evaluates retirement systems in the light of their capacity to make the short-lived better off.

The ethical justification for relying on the *ex post* egalitarian social welfare criterion in the present context is that longevity inequalities are here circumstances on which individuals have no control. Hence, on the basis of the Principle of Compensation (Fleurbaey and Maniquet 2004, Fleurbaey 2008), those arbitrary inequalities due to circumstances should be compensated by governments.

Under the *ex post* egalitarian social welfare criterion, the social planner chooses $\left\{c, d, e, \ell, \tilde{\ell}, K\right\}$ that maximize the realized lifetime well-being of the worst off living at the stationary equilibrium, subject to the resource constraint of the economy. That problem can be written as:

$$
\begin{aligned}
\max_{c,d,e,\ell,\tilde{\ell},K} \ \min \quad & \left\{u\left(c\right)-v\ell, u\left(c\right)-v\ell+u(d)-\tilde{v}\tilde{\ell}, u\left(c\right)-v\ell+u(d)-\tilde{v}\tilde{\ell}+u(e)\right\} \\
\text{s.t.} \quad & F\left(K, aN\ell+b\pi N\tilde{\ell}\right)=Nc+\pi Nd+p\pi Ne+K \\
\text{s.t.} \quad & \ell \geq 0 \text{ and } 1-\ell \geq 0 \\
\text{s.t.} \quad & \tilde{\ell} \geq 0 \text{ and } 1-\tilde{\ell} \geq 0
\end{aligned}
$$

That planning problem is not analytically tractable, since the $min\left(\cdot\right)$ function is not continuous. It is thus convenient to rewrite that planning problem as the maximization of the well-being of the short-lived, subject to the resource constraint, and subject to egalitarian constraints, which specify that the short-lived is, *ex post*, neither worse-off nor better off than long-lived individuals, whatever these reach the very old age or not. That planning problem is:

$$
\begin{array}{lrl}
\max\limits_{c,d,e,\ell,\tilde{\ell},K} & & N\left[u\left(c\right)-v\ell\right] \\
\text{s.t.} & & F\left(K,aN\ell+b\pi N\tilde{\ell}\right)=Nc+\pi Nd+p\pi Ne+K \\
\text{s.t.} & & u(d)-\tilde{v}\tilde{\ell}=0 \\
\text{s.t.} & & u(e)=0 \\
\text{s.t.} & \ell\geq & 0 \text{ and } 1-\ell\geq 0 \\
\text{s.t.} & \tilde{\ell}\geq & 0 \text{ and } 1-\tilde{\ell}\geq 0
\end{array}
$$

where $u(d)-\tilde{v}\tilde{\ell}=0$ is the egalitarian constraint for the old age, which specifies that individuals reaching the old age are neither better off nor worse off than the ones who do not reach the old age, whereas $u(e)=0$ is the egalitarian constraint for the very old age, which specifies that individuals reaching the very old age are as well-off as the ones who do not reach the very old age.

That social planning problem is solved in the Appendix. Proposition 5 summarizes our main results.[16]

**Proposition 5** *Consider the long-run ex post egalitarian optimum* $\left\{c^e,d^e,e^e,\ell^e,\tilde{\ell}^e,K^e\right\}$. *Define* $\mu\equiv\frac{\pi Nu'(c^e)}{u'(d^e)}$ *as the shadow value of relaxing the old-age egalitarian constraint.*

- *If* $\frac{v}{a}<\frac{\mu\tilde{v}}{\pi Nb}$, *then standard retirement holds* ($\tilde{\ell}^e=0$), *and we have:*

$$
\begin{array}{rcl}
c^e & > & d^e=\bar{c}=e^e \\
u'(c^e)F_L\left(K^e,aN\ell^e\right) & \geq & \dfrac{v}{a} \text{ and } F_K\left(K^e,aN\ell^e\right)=1
\end{array}
$$

- *If* $\frac{v}{a}>\frac{\mu\tilde{v}}{\pi Nb}$, *then reverse retirement prevails* ($\ell^e=0$), *and we have:*

$$
\begin{array}{rcl}
c^e & > & d^e=u^{-1}(\tilde{v}\tilde{\ell}^e)>e^e=\bar{c} \\
u'(c^e)F_L\left(K^e,b\pi N\tilde{\ell}^e\right) & \geq & \dfrac{\mu\tilde{v}}{\pi Nb} \text{ and } F_K\left(K^e,b\pi N\tilde{\ell}^e\right)=1
\end{array}
$$

**Proof.** See the Appendix. ■

Proposition 5 states that whether standard retirement or reverse retirement is part of the *ex post* egalitarian optimum depends on whether labor productivity increases or decreases with age (i.e. $a\gtreqless b$), and on the extent to which the disutility of labor increases with age (i.e. $v<\tilde{v}$). But the conditions under which standard or reverse retirement prevails differ from the ones prevailing under the utilitarian social criterion (Proposition 3). Quite interestingly, it is possible that reverse retirement is optimal from an *ex post* egalitarian perspective *even if* old-age labor is less productive than young-age labor, that is, even if $a>b$. That case was clearly not possible under the utilitarian criterion.

[16] We omit the case where $\frac{v}{a}=\frac{\mu\tilde{v}}{\pi Nb}$, in which the optimal working period is indeterminate.

The underlying intuition is that it may still be the case that making an old work brings, at the margin, less output and creates more disutility, but that can nonetheless be optimal under some conditions, since the *ex post* egalitarian optimum focuses only on the well-being of the worst-off, who is, in general, the short-lived. From that perspective, it is possible that maximal well-being at the young age - and, thus, for the unlucky short-lived - is achieved by making the young retired (even though this may reduce aggregate consumption possibilities with respect to standard retirement).

Actually, whether standard or reverse retirement prevails at the *ex post* egalitarian optimum depends on the precise form of the temporal utility of consumption $u(\cdot)$, through the key role played by the shadow value of relaxing the old-age egalitarian constraint.[17] The intuition behind the key role of $u(\cdot)$ comes from the egalitarian constraint concerning the old. That constraint specifies that, at the *ex post* egalitarian optimum, the utility of old-age consumption $d$ must exactly compensate the disutility of old-age labor $\tilde{v}\tilde{\ell}$, so that $u(d) = \tilde{v}\tilde{\ell}$. As a consequence of that egalitarian constraint, the curvature of $u(\cdot)$ determines, for a given $\tilde{\ell}$, the amount of consumption that is required for compensation.

Although none of the two cases of Proposition 5 can be excluded *a priori*, there are strong reasons to believe that reverse retirement is likely to be part of the *ex post* egalitarian optimum, provided the disutility of old-age labor is not too large, and provided old-age productivity is sufficiently large with respect to young-age productivity. The intuition goes as follows. The old-age egalitarian constraint leads to low old-age consumption levels, at which $u'(d)$ is large. Hence, the shadow value of relaxing the old-age egalitarian constraint $\mu = \frac{\pi N u'(c^e)}{u'(d^e)}$ is low, so that it is quite likely that $\frac{v}{a} > \frac{\mu \tilde{v}}{\pi N b}$, which implies reverse retirement. Thus there are reasons to believe that the *ex post* egalitarian optimum involves, in many cases, reverse retirement.

Having stressed this, it is hard to have a certainty about the prevailing case without imposing particular functional forms for the production technology and for preferences. Proposition 6 derives the conditions under which reverse retirement or standard retirement prevails at the *ex post* egalitarian optimum, under a logarithmic utility function and a Cobb-Douglas production function.

**Proposition 6** *Assume* $u(c_t) = \log(c_t) - \beta$ *and* $Y_t = AK_t^{\alpha}\left(aN\ell_t + b\pi N\tilde{\ell}_t\right)^{1-\alpha}$ *with* $0 < \alpha < \frac{1}{2}$. *Assume that* $\max\left\{\ell, \tilde{\ell}\right\} < 1$. *Define* $\bar{c} = \exp(\beta)$ *and* $\Xi \equiv A(1-\alpha)(A\alpha)^{\frac{\alpha}{1-\alpha}}$, *as well as* $\Phi \equiv \log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta - 1$. *Define also:*

$$\eta \equiv \log\left(\frac{a}{v}\Xi\right) - \beta - 1 - v\frac{\pi(1+p)\bar{c}}{a\Xi} \text{ and } \xi \equiv \log\left(\frac{b\pi}{\tilde{v}}\Xi\Phi - \pi p\bar{c}\right) - \beta$$

- *If* $\max\{\eta, \xi\} = \eta$, *then the long-run ex post egalitarian optimum involves*

[17] That shadow value is not a parameter, but a variable, which depends on the optimal values $\left\{c^e, d^e, e^e, \ell^e, \tilde{\ell}^e, K^e\right\}$. Proposition 6 derives conditions based only on parameters, under which either standard or reverse retirement prevails at the *ex post* egalitarian optimum.

*standard retirement (i.e.* $\tilde{\ell}^e = 0$*), and we have:*

$$c^e = \frac{a\Xi}{v};\ \ell^e = \frac{1 + \frac{v\bar{c}}{a\Xi}\pi(1+p)}{v};\ K^e = aN\left(A\alpha\right)^{\frac{1}{1-\alpha}} \frac{\left(1 + \frac{v\pi(1+p)\bar{c}}{a\Xi}\right)}{v}$$

- *If* $\max\{\eta, \xi\} = \xi$*, then the long-run ex post egalitarian optimum involves reverse retirement (i.e.* $\ell^e = 0$*), and we have:*

$$c^e = \frac{b\Xi}{\tilde{v}}\pi\Phi - \pi p\bar{c};\ \tilde{\ell}^e = \frac{1+\Phi}{\tilde{v}};\ K^e = \pi bN\left(A\alpha\right)^{\frac{1}{1-\alpha}} \frac{\pi\left(1+\Phi\right)}{\pi\tilde{v}}$$

**Proof.** See the Appendix. ■

Proposition 6 identifies parameter restrictions under which the *ex post* egalitarian optimum involves either standard or reverse retirement. As such, Proposition 6 cast some light on the conditions under which reverse retirement is optimal from an *ex post* egalitarian perspective. Take, for instance, the role played by survival conditions $\{\pi, p\}$. When the survival probability $\pi$ is lower, $\eta$ is larger, whereas $\xi$ is, in general, lower, so that a lower $\pi$ favors standard retirement. On the contrary, a higher $\pi$ favors reverse retirement. Note, however, that the role of the (conditional) probability of survival to the very old age $p$ is ambiguous, since both $\eta$ and $\xi$ are decreasing in $p$.

One can use Proposition 6 to compare traditional economies and modern economies, which differ on labor productivity $\{a, b\}$, on disutility of labor $\{v, \tilde{v}\}$ and on survival conditions $\{\pi, p\}$.[18] In traditional economies, working conditions are harsh, leading to a high $\frac{\tilde{v}}{v}$, and labor productivity strongly declines with age (because of the highly physical nature of labor), leading to $a >> b$. Moreover, the probability $\pi$ to survive to age 50 is low. In traditional economies, it is thus likely that $\max\{\eta, \xi\} = \eta$, so that standard retirement is optimal. However, in modern economies, working conditions are less harsh, leading to a lower $\frac{\tilde{v}}{v}$, labor productivity is more stable with age, leading to $a \simeq b$, and $\pi$ is larger. Hence, in modern economies, it is more likely that $\max\{\eta, \xi\} = \xi$, i.e., reverse retirement is optimal from an *ex post* egalitarian perspective.

In sum, this section shows that whether reverse retirement can be socially optimal or not is not robust to the postulated social welfare criterion. Under a utilitarian social objective, standard retirement is generally part of the social optimum, and reverse retirement is, in general, not optimal (except if old workers are much more productive than young workers, which is a strong assumption). However, if one gives priority to the worst-off *ex post*, then reverse retirement can, in some cases, be part of the social optimum. In particular, reverse retirement can, in some cases, be socially optimal even when old workers are less productive than young workers, *unlike what prevails under utilitarianism.*

[18] On structural differences between traditional and modern economies, see North (1981).

# 5 Decentralization

Up to now, we compared economies with standard and reverse retirement, and examined which type of retirement is socially optimal under different social welfare criteria. At this stage, one may be curious to know how a government could, by using appropriate policy instruments, make the decentralized economy converge towards the long-run social optimum.

For that purpose, we will restrict ourselves to the general case where there is underaccumulation of capital. Moreover, we will also suppose that standard retirement prevails at the laissez-faire ($\tilde{\ell} = 0$), so that we have:[19]

$$\begin{aligned} u'\left(c^{LF}\right) &= Ru'\left(d^{LF}\right) && (13) \\ u'\left(d^{LF}\right) &= Ru'\left(e^{LF}\right) && (14) \\ wu'\left(c^{LF}\right) &= v && (15) \end{aligned}$$

where wages and interest rates satisfy, in the competitive economy,

$$\begin{aligned} w &= aF_L\left(K^{LF}, aN\ell^{LF}\right) && (16) \\ R &= F_K\left(K^{LF}, aN\ell^{LF}\right) && (17) \end{aligned}$$

This section examines how the government, acting as a Stackelberg leader, can use appropriate taxes and transfers to decentralize the social optimum.

## 5.1 Decentralization of the utilitarian optimum

Let us first consider the decentralization of the utilitarian social optimum. We know, from above, that when young workers are weakly more productive than old workers, the long-run utilitarian optimum involves standard retirement, as in the laissez-faire. Remind that, at the utilitarian optimum with standard retirement, we have:[20]

$$\begin{aligned} u'(c^u) &= u'(d^u) = u'(e^u) && (18) \\ aF_L\left(K^u, aN\ell^u\right) u'(c^u) &= v && (19) \\ F_K\left(K^u, aN\ell^u\right) &= 1 && (20) \end{aligned}$$

Proposition 7 summarizes our results concerning the decentralization of the long-run utilitarian optimum.

**Proposition 7** *The long-run utilitarian optimum* $\left\{c^u, d^u, e^u, \ell^u, \tilde{\ell}^u, K^u\right\}$ *with standard retirement can be decentralized by means of an intergenerational lump-sum transfer device leading to a capital stock* $K = K^u$ *such that:*

$$\begin{aligned} F_K\left(K^u, aN\ell^u\right) &= 1 \\ F_L\left(K^u, aN\ell^u\right) u'\left(c^u\right) &= v \end{aligned}$$

[19] We assume here that $\ell$ is an interior solution, i.e. $\ell < 1$.

[20] We assume here that $\ell^u$ is an interior solution, i.e. $\ell^u < 1$.

**Proof.** See the Appendix. ■

The decentralization of the long-run utilitarian optimum requires only one instrument: intergenerational lumpsum transfers leading to the Golden Rule capital stock (Phelps 1961). The intuition is that the government's objective differs only from individual's objective as far as the time horizon is concerned, but not on other aspects. Individuals save resources while taking their own lifetime horizon into account, whereas the government considers the saving that maximizes utility at the steady-state. Hence, the government must impose intergenerational lumpsum transfers that decentralize the optimal capital stock. But once the capital stock is optimal, individuals choose, in a competitive economy, retirement ages and consumption profiles that are socially optimal.

Undoubtedly, the fact that the laissez-faire equilibrium already involves standard retirement facilitates the decentralization of the long-run utilitarian optimum, since the decentralization does not require a shift from one retirement system to another. As we will see below, things are less simple when considering the decentralization of the *ex post* egalitarian optimum when this requires to shift from standard to reverse retirement.

## 5.2 Decentralization of the ex post egalitarian optimum

Since there are good reasons to believe that the *ex post* egalitarian optimum involves, in advanced economies, reverse retirement, we will, throughout this section, examine the decentralization problem of the *ex post* egalitarian optimum with reverse retirement, and leave the other case aside.

That decentralization problem raises particular difficulties. An important difficulty lies in the fact that, at the laissez-faire, we have, under general conditions, standard retirement. Hence the decentralization of the *ex post* egalitarian optimum requires nothing less than converting an economy with standard retirement into an economy with reverse retirement.

To study that decentralization problem, remind that, at the *ex post* egalitarian optimum with reverse retirement, we have:[21]

$$c^e > d^e = u^{-1}(\tilde{v}\tilde{\ell}^e) > e^e = \bar{c} \tag{21}$$

$$F_L\left(K^e, b\pi N\tilde{\ell}^e\right) u'(c^e) = \frac{\mu\tilde{v}}{\pi N b} \tag{22}$$

$$F_K\left(K^e, b\pi N\tilde{\ell}^e\right) = 1 \tag{23}$$

Proposition 8 summarizes our results.

**Proposition 8** *The long-run ex post egalitarian optimum* $\left\{c^e, d^e, e^e, \ell^e, \tilde{\ell}^e, K^e\right\}$ *with reverse retirement can be decentralized by means of:*

- *a prohibition of young-age labor:* $\ell = \ell^e = 0$*;*
- *a legal retirement age fixed at* $2 + \tilde{\ell} = 2 + \tilde{\ell}^e$*;*

[21] We assume here that $\tilde{\ell}^e$ is an interior solution, i.e. $\tilde{\ell}^e < 1$.

- *a subsidy $\theta$ on young-age borrowing satisfying:* $\theta = \frac{u'(d^e)}{u'(c^e)} - 1 > 0$*;*

- *a tax $\tau$ on old-age savings satisfying:* $\tau = 1 - \frac{u'(d^e)}{u'(e^e)} > 0$*;*

- *an intragenerational lumpsum transfer device leading to the egalitarian constraint at the old age:* $T^e = d^e - d^{LF}$*;*

- *an intragenerational lumpsum transfer device leading to the egalitarian constraint at the very old age:* $\tilde{T}^e = e^e - e^{LF}$*;*

- *an intergenerational lumpsum transfer device leading to a capital stock* $K = K^e$ *such that:*
$$F_K\left(K^e, \pi b N \tilde{\ell}^e\right) = 1.$$

**Proof.** See the Appendix. ■

The decentralization of the *ex post* egalitarian optimum requires not less than 7 instruments. This large number of instruments is justified by the sizeable departure between the laissez-faire equilibrium and the *ex post* egalitarian optimum. Under the former, individuals work when being young, save resources for the old age, during which they are retired. On the contrary, under the latter, individuals do not work when being young, and work during the old age (reverse retirement). Moreover, the consumption profile prevailing at the laissez-faire differs from the socially optimal one, and this difference motivates the use of a subsidy on young-age borrowing, a tax on old-age savings, as well as lump-sum transfers guaranteeing that the egalitarian constraints are satisfied at the old age and the very old age. All those differences explain why the decentralization of the *ex post* egalitarian optimum requires more instruments than the decentralization of the utilitarian optimum.

It is important to highlight that the policy instruments in Proposition 8 allow the government to overcome the difficulties raised by the *transition* from one retirement regime to another regime. As we underlined above, the laissez-faire economy could hardly bear a transition from standard retirement to reverse retirement, because such a transition would be characterized by the absence of workers on the labor market during the period of transition. Indeed, if the transition takes place at time $t$, young individuals at $t$ plan to work only at the old age, which takes place at $t+1$, whereas old individuals at $t$ have already worked when being young, and do not plan to work at the old age. As a consequence, in the laissez-faire, the transition from standard retirement to reverse retirement leads to the collapse of the economy, because of the absence of labor (and of production) during one period of time.

This transition problem is avoided by the government in Proposition 8, which includes, among the instruments, the fixation of a legal retirement age at age $2+\tilde{\ell}^e$. This instrument forces individuals who are old at the time of the transition (and who worked when being young) to work also during the transition period, which allows the economy to produce output despite the retirement of young adults. Thus fixing a compulsory legal retirement age at age $2+\tilde{\ell}^e$ prevents the old from not working, and, hence, prevents the collapse of the economy during the transition period.

In sum, although a laissez-faire economy could not exhibit a transition from standard to reverse retirement without collapsing, public policy can allow for that transition. This transition has a cost for a transition cohort, who has to work both at the young age (before the policy is implemented) and at old age (when the policy is implemented). But that cost is necessary to avoid the collapse of the economy. The policies mentioned in Proposition 8 are thus able to overcome the transition problems prevailing at the laissez-faire.

# 6 Discussions

Previous sections derived positive results concerning reverse retirement. First, on the positive side, an economy with reverse retirement converges towards a unique stationary equilibrium with a strictly positive capital stock. Second, on the normative side, reverse retirement can be part of the *ex post* egalitarian optimum even when old workers are less productive than young ones (unlike under the utilitarian criterion). Moreover, there exist policy instruments that decentralize the *ex post* egalitarian optimum with reverse retirement.

This section considers some criticisms against reverse retirement, both on the normative side (social desirability) and on the positive side (feasibility).

## 6.1 Reward and efforts

A first line of criticism concerns the ethical foundations of reverse retirement. As shown above, reverse retirement can be justified, under some conditions, as part of the *ex post* egalitarian optimum, which does justice to the idea of compensating the unlucky short-lived. From that perspective, reverse retirement draws its justification from the Principle of Compensation.

One may reply to this that the Principle of Compensation is only one ethical principle among many others. In particular, one may argue that a fair retirement system should be based not on the Principle of Compensation, but on the Principle of Liberal Reward (see Fleurbaey and Maniquet, 2004, Fleurbaey, 2008). According to that principle, inequalities due to individual efforts should be left unaffected by governments.

The Principle of Liberal Reward can be used to justify standard retirement. From that perspective, standard retirement would provide a fair reward for a working career. Only individuals who made efforts (i.e. who worked) should be rewarded by a retirement period. Therefore, replacing standard retirement by reverse retirement would be unfair, since this would give rise to a "free lunch" for individuals who turn out to die before starting to work.

To examine that criticism, it should be first underlined that the Principle of Compensation and the Principle of Liberal Reward are compatible *only if* circumstances and effort variables do not interact in the production of individual outcomes (Fleurbaey and Maniquet, 2004, Fleurbaey, 2008). However, in our setting, inequalities in realized well-being depend *both* on how long individuals work (i.e. efforts), and on how long they live (i.e. circumstances). Hence, it is logically impossible to satisfy both principles in our context.

As a consequence, a choice is to be made, at the ethical level, between the Principle of Compensation and the Principle of Liberal Reward. It is true that, under reverse retirement, prematurely dead individuals benefit from a kind of "free lunch" (i.e. they consume resources without working). This is somewhat unfair with respect to individuals surviving to the old age, who will have to work and pay back their debt. But under standard retirement, prematurely dead individuals work and then die, without having benefited from retirement. This "no reward" situation is also unfair.

There are some reasons to think that the "no reward" situation is more unfair than the "free lunch" enjoyed by the short-lived under reverse retirement. The intuition goes as follows. A "free lunch" for the prematurely dead can be justified as a kind of compensation for a serious damage. On the contrary, there is nothing that can justify the absence of any reward for workers who turn out to die prematurely. Those unlucky short-lived individuals face a double penalty: first, these have a shorter life, and second, the standard retirement system leaves the "good things" for the end of life. This double penalty is more unfair than the "free lunch" enjoyed under reverse retirement. All in all, the "reward and effort" argument is not successful at ruling out reverse retirement.

## 6.2 The insurance motive

A second line of criticism is rooted in another common argument supporting standard retirement. Actually, among economists, the main justification for the standard retirement system would not be the reward for a long working career, but, rather, the insurance motive.

The insurance motive for standard retirement system, studied by Barr and Diamond (2006) and Cremer and Pestieau (2011), goes as follows. Individuals, when being young workers, tend to be myopic, and tend to save too few resources for their old age. As a consequence, individuals take the risk of being poor in case of a long life. The main task of a standard retirement system is then, by providing pensions at the old age, to insure individuals against old-age poverty.

In the light of this, it is tempting to claim that replacing standard retirement by reverse retirement would not be desirable, because this would go against that insurance motive, which is the major justification for retirement systems.

However, quite interestingly, that criticism does not weaken, but tends to reinforce, the support for a reverse retirement system.

Clearly, the above criticism examines the insurance motive while focusing on the risk of old-age poverty, whereas, from the perspective of lifetime well-being, a more substantial risk consists of the risk of a short life. Indeed, the welfare loss associated to a short life is, under general conditions, larger than the welfare loss associated to old-age poverty. Therefore, if one really cares about insuring individuals, the priority should be insure them against a short life. This is precisely what the reverse retirement system does: reverse retirement minimizes well-being losses due to the occurrence of a short life. Thus this alternative retirement system relies also on an insurance motive, but covers a risk that is more substantial than the one covered by standard retirement.

### 6.3 Productivity and learning by doing

A third line of criticism lies on the positive side, and concerns the impact of reverse retirement on labor productivity, on the potential for economic growth, and, hence, for consumption possibilities. In particular, one may argue that postponing the entrance on the labor market may prevent individuals from acquiring a strong experience in the firm. Since Adam Smith (1776)'s pin factory example, it has been argued by economists that a major source of productivity growth lies in workers' repetition of actions. By repeating their actions, workers become more and more productive. This is close to the idea of "learning by doing" by Arrow (1962). In the light of this, adopting a reverse retirement system may prevent the economy from enjoying a substantial learning by doing, and, hence, may make significant productivity gains vanish.

Although that criticism is relevant for the issue at stake, it should be stressed that this argument is far from decisive. That argument is true only to the extent that learning by doing can be made *only* at the young age. However, from the perspective of allowing for repeated activity and learning about the production process, the reverse retirement system does not seem to prevent those processes, and the associated productivity gains.

But even if one assumes that "learning by doing" is more difficult for older workers than for younger workers, this assumption can be taken into account by assuming $a >> b$ in our model. However, assuming a strong productivity gap is not sufficient to rule out reverse retirement under the *ex post* egalitarian social criterion. As shown above, it is possible, under some conditions, that reverse retirement is part of the *ex post* egalitarian optimum despite old workers being less productive than young workers.

### 6.4 The transition

A fourth line of criticism concerns not the economic feasibility of the reverse retirement system, but the feasibility of the transition from standard retirement to reverse retirement. As stated in Proposition 8, the decentralization of the *ex post* egalitarian optimum with reverse retirement requires to impose a legal retirement age at age $2 + \tilde{\ell}^e$, which forces a particular cohort to work not only during the young age (i.e. before the policy is implemented), but, also, during the old age (i.e. after the policy is implemented). Imposing such a constraint on the transition cohort is necessary to avoid the collapse of the economy (absence of labor in one period).

Undoubtedly, the particular transition cohort (individuals who are young adults before the transition, and old adults after the transition) faces the inconvenient of having no retirement at all (except at the very old age). Those individuals have to work both at the young age and at the old age. That transition cohort is thus likely to be strongly against the implementation of reverse retirement. One can thus expect strong resistance from that particular cohort. This could constitute a sizeable obstacle to reverse retirement.

However, one should not overestimate the strength of that obstacle. Actually, our 4-period OLG economy is a reduced-form model, where the entire burden of the transition lies on the shoulders of that transition cohort. But in real-world economies, the labor force is not made of 2 cohorts, but of 45 cohorts. It is thus possible, in real-world economies, to smooth the transition from one retirement regime to another, without imposing a strong burden on a single cohort. The transition from standard to reverse retirement could be actually quite smooth, with a progressive postponement of ages of entry in the labor market, and of ages of exit from the labor market.

# 7 Concluding remarks

Under unequal lifetimes, the standard retirement system, in which individuals work a long life before enjoying retirement, does not look fair, since it implies that some unlucky individuals work and die before enjoying retirement. But would the - purely hypothetical - reverse retirement system (in which individuals are first retiree and then work) be more fair to the unlucky short-lived?

This paper proposed to reexamine the fairness of retirement systems by comparing standard retirement with reverse retirement. Our analysis leads us to three main results, which concern the economic feasibility of reverse retirement, as well as its social desirability.

First of all, concerning the feasibility of reverse retirement, we showed that, under standard assumptions on the production technology and on preferences, an economy with reverse retirement - once in place - converges, in the long-run, towards a unique stationary equilibrium with a strictly positive capital stock. That positive result is important, especially since one may believe, at first glance, that reverse retirement is a social utopia that would not be sustainable in the long-run. This is clearly not the case.

Second, our analysis shows that the social desirability of reverse retirement depends on the underlying ethical foundations. Under the utilitarian criterion, reverse retirement cannot be a social optimum (unlike standard retirement) when labor productivity declines with age. However, if one adopts the *ex post* egalitarian criterion (giving priority to the worst-off *ex post*, who is, in general, the short-lived), then, it can be the case, under some conditions, that reverse retirement is a social optimum (even when labor productivity declines with age). But even if one adopts the *ex post* egalitarian criterion, it is not necessarily the case that reverse retirement dominates standard retirement. Actually, in less developed countries (with a low survival probability to age 50 and a steep age-productivity profile), standard retirement dominates reverse retirement, even from the perspective of the well-being of the unlucky short-lived. On the contrary, in advanced economies, where production does not require high physical efforts (leading to a lower age productivity gap), and where there is a higher survival probability to age 50, reverse retirement dominates standard retirement. Therefore the *ex post* egalitarian argument supporting a shift from standard to reverse retirement holds only for sufficiently advanced economies.

Third, although the transition from standard to reverse retirement would lead, at the laissez-faire, to the collapse of the economy (due to the absence of labor in one period), our analysis of the decentralization of the *ex post* egalitarian optimum with reverse retirement shows that there exists a set of policy instruments that allow governments to organize a successful transition from standard to reverse retirement. Thus the design of adequate policy instruments allows to overcome the transition problems faced at the laissez-faire.

It is important to stress here that the *ex post* egalitarian argument supporting a reverse retirement system is distinct from other possible arguments. A first alternative argument could be based on education and human capital accumulation. Reverse retirement could stimulate investment in education, which would favor economic growth. Our argument differs from this, since it involves neither education choices, nor assumptions on the return of education. Another argument would consist in claiming that reverse retirement would allow young people to work for pro-social NGOs in a benevolent way, in the same way as retirees give their time to NGOs nowadays. But this differs from our argument, which does not require any pro-social sector.

To conclude, although this study derived positive results regarding the feasibility and the social desirability of reverse retirement, it highlighted also some problems, concerning the transition from standard to reverse retirement. Such a transition would, at the laissez-faire, lead the economy to collapse. Fortunately, public policy could be used to organize the transition without any collapse, but at the cost of requiring a transition cohort to work both at the young age and at the old age. Although the transition could be, in real-world economies, slow (i.e. progressive postponement of entry in and exit from the labor market), the shift from standard to reverse retirement may face strong resistances. Note, however, that, during the last centuries, there were already sizeable postponements of entry on the labor market (end of child labor in industrialized economies) as well as significant postponements of exit from the labor market. Thus reverse retirement may not be as utopian as it may look at first glance.

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# 9 Appendix

## 9.1 Proof of Proposition 1

The problem of the individual can be written by means of the Lagrangian:

$$\begin{aligned}\max_{s_t, z_{t+1}, \ell_t, \tilde{\ell}_{t+1}} \quad & u\left(w_t \ell_t - s_t\right) - v\ell_t + \pi\left[u\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) - \tilde{\ell}_{t+1}\tilde{v}\right] \\ & + \pi p u\left(\frac{R_{t+2}^{E_t} z_{t+1}}{p}\right) + \rho\ell_t + \varsigma(1-\ell_t) + \varphi\tilde{\ell}_{t+1} + \psi(1-\tilde{\ell}_{t+1})\end{aligned}$$

FOCs are

$$\begin{aligned} u'\left(w_t\ell_t - s_t\right) &= R_{t+1}^{E_t} u'\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) \\ u'\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) &= R_{t+2}^{E_t} u'\left(\frac{R_{t+2}^{E_t} z_{t+1}}{p}\right) \\ w_t u'\left(w_t\ell_t - s_t\right) &= v - \rho + \zeta \\ \tilde{w}_{t+1}^{E_t} u'\left(\tilde{w}_{t+1}^{E_t}\tilde{\ell}_{t+1} + \frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) &= \tilde{v} - \varphi + \psi \end{aligned}$$

as well as conditions

$$\begin{aligned} \rho &\geq 0, \ell_t \geq 0 \text{ and } \varsigma \geq 0, 1-\ell_t \geq 0 \\ \varphi &\geq 0, \tilde{\ell}_{t+1} \geq 0 \text{ and } \psi \geq 0, 1-\tilde{\ell}_{t+1} \geq 0 \end{aligned}$$

with complementary slackness.

Substituting the first FOC in the fourth one, we obtain:

$$u'\left(w_t\ell_t - s_t\right) = \frac{v-\rho+\zeta}{w_t} = R_{t+1}^{E_t}\frac{\tilde{v}-\varphi+\psi}{\tilde{w}_{t+1}^{E}}$$

If $\frac{v}{w_t} < \frac{R_{t+1}^{E_t}\tilde{v}}{\tilde{w}_{t+1}^{E_t}}$, then we have $\tilde{\ell}_{t+1} = 0$, $\varphi > 0$ and $\psi = 0$. We also have $\rho = 0$ and $\varsigma \geq 0$. Thus standard retirement holds ($\tilde{\ell}_{t+1} = 0$), and we have:

$$\begin{aligned} u'\left(w_t\ell_t - s_t\right) &= R_{t+1}^{E_t} u'\left(\frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) \\ u'\left(\frac{R_{t+1}^{E_t} s_t}{\pi} - z_{t+1}\right) &= R_{t+2}^{E_t} u'\left(\frac{R_{t+2}^{E_t} z_{t+1}}{p}\right) \\ w_t u'\left(w_t\ell_t - s_t\right) &\geq v \end{aligned}$$

If $\frac{v}{w_t} > \frac{R^{E_t}_{t+1}\tilde{v}}{\tilde{w}^{E_t}_{t+1}}$, then we have $\ell_t = 0$, $\rho > 0$ and $\varsigma = 0$. We also have $\varphi = 0$ and $\psi \geq 0$. Hence reverse retirement holds ($\ell_t = 0$), and we have:

$$\begin{aligned} u'(-s_t) &= R^{E_t}_{t+1} u' \left( \tilde{w}^{E_t}_{t+1} \tilde{\ell}_{t+1} + \frac{R^{E_t}_{t+1} s_t}{\pi} - z_{t+1} \right) \\ u' \left( \tilde{w}^{E_t}_{t+1} \tilde{\ell}_{t+1} + \frac{R^{E_t}_{t+1} s_t}{\pi} - z_{t+1} \right) &= R^{E_t}_{t+2} u' \left( \frac{R^{E_t}_{t+2} z_{t+1}}{p} \right) \\ \tilde{w}^{E_t}_{t+1} u' \left( \tilde{w}^{E_t}_{t+1} \tilde{\ell}_{t+1} + \frac{R^{E_t}_{t+1} s_t}{\pi} - z_{t+1} \right) &\geq \tilde{v} \end{aligned}$$

as stated in Proposition 1.

## 9.2 Proof of Proposition 2

**Standard retirement** Under perfect foresight, savings at the young age and the old age satisfy, under our assumptions on $u(\cdot)$:

$$\begin{aligned} \frac{1}{w_t \ell_t - s_t} &= \frac{R_{t+1}}{\frac{R_{t+1} s_t}{\pi} - z_{t+1}} \\ \frac{1}{\frac{R_{t+1} s_t}{\pi} - z_{t+1}} &= \frac{R_{t+2}}{\frac{R_{t+2} z_{t+1}}{p}} \end{aligned}$$

From the first FOC:

$$\frac{R_{t+1} s_t}{\pi} - z_{t+1} = R_{t+1} (w_t \ell_t - s_t) \rightarrow s_t = \frac{R_{t+1} (w_t \ell_t)}{R_{t+1}(\frac{1}{\pi} + 1)} + \frac{z_{t+1}}{R_{t+1}(\frac{1}{\pi} + 1)}$$

From the second FOC:

$$z_{t+1} = \frac{p}{1+p} \frac{R_{t+1} s_t}{\pi}$$

Hence

$$s_t = \frac{(w_t \ell_t)}{(\frac{1}{\pi} + 1)} + \frac{\frac{p}{1+p} \left( \frac{R_{t+1} s_t}{\pi} \right)}{R_{t+1}(\frac{1}{\pi} + 1)} = \frac{\pi(1+p)}{(1 + \pi + p\pi)} (w_t \ell_t)$$

Hence

$$z_{t+1} = \frac{p}{(1 + \pi + p\pi)} (w_t \ell_t) R_{t+1}$$

Remind that $\ell_t$ is determined by (at interior):

$$\frac{w_t}{w_t \ell_t - s_t} = v \rightarrow \ell_t = \frac{1}{v} + \frac{s_t}{w_t}$$

Hence savings can be rewritten as:

$$s_t = \frac{\pi(1+p)}{(1+\pi+p\pi)}\left(w_t\left(\frac{1}{v}+\frac{s_t}{w_t}\right)\right) = \pi(1+p)\left(\frac{w_t}{v}\right)$$

Hence old-age savings is:

$$z_{t+1} = \frac{p}{1+p}\left(\frac{R_{t+1}\pi(1+p)\left(\frac{w_t}{v}\right)}{\pi}\right) = pR_{t+1}\left(\frac{w_t}{v}\right)$$

Hence:

$$\ell_t = \frac{1}{v} + \frac{\pi(1+p)\left(\frac{w_t}{v}\right)}{w_t} = \frac{1+\pi(1+p)}{v}$$

The accumulation equation is:

$$K_{t+1} = Ns_t + N\pi z_t = N\left[\pi(1+p)\left(\frac{w_t}{v}\right)\right] + N\pi\left[pR_t\left(\frac{w_{t-1}}{v}\right)\right]$$

Note that we have: $w_t = aF_L = a(1-\alpha)AK_t^{\alpha}(a\ell_t)^{-\alpha} = a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}$ and $R_t = F_K = \alpha AK_t^{\alpha-1}(a\ell_t)^{1-\alpha} = \alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}$. Hence the capital accumulation equation is:

$$\begin{aligned} K_{t+1} &= N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right] \\ &\quad + N\pi\left[p\alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\left(\frac{a(1-\alpha)AK_{t-1}^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right] \end{aligned}$$

We have a one-dimensional dynamic system with two time lags.

By defining the variable $\omega_t \equiv \frac{a(1-\alpha)AK_{t-1}^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}$, the system can be rewritten as a dynamic system with one time lag but two variables:

$$\begin{aligned} K_{t+1} &= \left[\begin{array}{l} N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right] \\ +N\pi\left[p\alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\omega_t\right] \end{array}\right] \\ \omega_{t+1} &= \left(\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right) \end{aligned}$$

The $KK$ locus is defined as the set of pairs $(K_t, \omega_t)$ such that $K_{t+1} = K_t$. It is defined as the relation:

$$K_t - N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right] = N\pi\left[p\alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\omega_t\right]$$

Hence

$$\omega_t = \frac{K_t^{2-\alpha} - N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK_t\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right]}{N\pi\left[p\alpha A\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\right]} \equiv G(K_t)$$

The $\Omega\Omega$ locus is defined as the set of pairs $(K_t, \omega_t)$ such that $\omega_{t+1} = \omega_t$. It is defined as the relation:

$$\omega_t = \left(\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right) \equiv H(K_t)$$

We have that $G(0) = 0$ and $H(0) = 0$. Hence $(0,0)$ is a stationary equilibrium.

Regarding the existence of a stationary equilibrium with $K_t > 0, \omega_t > 0$, note that such an equilibrium exists when we have, for some $K_t > 0$, $H(K_t) = G(K_t)$, that is, when:

$$\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v} + \frac{(1+p)\frac{a(1-\alpha)K_t}{v}}{p\alpha\left(a\frac{1+\pi(1+p)}{v}\right)} = \frac{K_t^{2-\alpha}}{N\pi\left[p\alpha A\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\right]}$$

Let us denote the LHS by $\Theta(K_t)$ and the RHS by $\Gamma(K_t)$. We have that

$$\Theta'(K_t) = \frac{a(1-\alpha)A\alpha K_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v} + \frac{(1+p)\frac{a(1-\alpha)}{v}}{p\alpha\left(a\frac{1+\pi(1+p)}{v}\right)} > 0$$

$$\Gamma'(K_t) = \frac{(2-\alpha)K_t^{1-\alpha}}{N\pi\left[p\alpha A\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\right]} > 0$$

Note also that:

$$\Theta''(K_t) = \frac{a(1-\alpha)A\alpha(\alpha-1)K_t^{\alpha-2}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v} < 0$$

$$\Gamma''(K_t) = \frac{(2-\alpha)(1-\alpha)K_t^{-\alpha}}{N\pi\left[p\alpha A\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\right]} > 0$$

Thus $\Theta(K_t)$ is increasing and concave, while $\Gamma(K_t)$ is increasing and convex.

We have also that, in the neighborhood of $(0,0)$:

$$\lim_{K_t\to 0}\Theta'(K_t) = \frac{a(1-\alpha)A\alpha K_t^{\alpha-1}(a\frac{1+\pi(1+p)}{v})^{-\alpha}}{v} + \frac{(1+p)\frac{a(1-\alpha)}{v}}{p\alpha(a\frac{1+\pi(1+p)}{v})} = +\infty$$

$$\lim_{K_t\to 0}\Gamma'(K_t) = \frac{(2-\alpha)K_t^{1-\alpha}}{N\pi\left[p\alpha A(a\frac{1+\pi(1+p)}{v})^{1-\alpha}\right]} = 0$$

Hence the function $\Theta(K_t)$ lies strictly above $\Gamma(K_t)$ in the neighborhood of $(0,0)$.

Note also that:

$$\begin{aligned}
\lim_{K_t\to+\infty}\Theta'(K_t) &= \frac{a(1-\alpha)A\alpha K_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}+\frac{(1+p)\frac{a(1-\alpha)}{v}}{p\alpha\left(a\frac{1+\pi(1+p)}{v}\right)}=\frac{(1+p)\frac{a(1-\alpha)}{v}}{p\alpha\left(a\frac{1+\pi(1+p)}{v}\right)}\\
\lim_{K_t\to+\infty}\Gamma'(K_t) &= \frac{(2-\alpha)K_t^{1-\alpha}}{N\pi\left[p\alpha A\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\right]}=+\infty
\end{aligned}$$

Hence the function $\Theta(K_t)$ lies strictly below $\Gamma(K_t)$ for large levels of $K_t$.

Hence, by continuity of $\Theta(K_t)$ and $\Gamma(K_t)$, there must exist at least one level of $K_t>0$ such that $\Theta(K_t)=\Gamma(K_t)$, which implies the existence of stationary equilibrium with $K_t>0,\omega_t>0$. By the monotonicity of $\Theta(K_t)$ and $\Gamma(K_t)$, and by the concavity of $\Theta(K_t)$ and the convexity of $\Gamma(K_t)$, that intersection is unique. There exists thus a unique strictly positive stationary equilibrium $K^{s**}>0$.

Let us now study the stability of the two stationary equilibria. We have:

$$\begin{aligned}
K_{t+1} &= \left[\begin{array}{ll} N\pi(1+p)\frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v} & \\ +N\pi p\alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha} & \omega_t\end{array}\right]\equiv f(K_t,\omega_t)\\
\omega_{t+1} &= \frac{a(1-\alpha)AK_t^{\alpha}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\equiv g(K_t)
\end{aligned}$$

The Jacobian matrix is defined as:

$$J=\begin{pmatrix}\frac{\partial f(\cdot)}{\partial K_t} & \frac{\partial f(\cdot)}{\partial \omega_t}\\ \frac{\partial g(\cdot)}{\partial K_t} & \frac{\partial g(\cdot)}{\partial \omega_t}\end{pmatrix}$$

We have

$$\begin{aligned}
\frac{\partial f(\cdot)}{\partial K_t} &= N\left[\pi(1+p)\left(\frac{a(1-\alpha)A\alpha K_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\right)\right]\\
&\quad +N\pi\left[p\alpha A(\alpha-1)K_t^{\alpha-2}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\omega_t\right]\\
\frac{\partial f(\cdot)}{\partial \omega_t} &= N\pi p\alpha AK_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{1-\alpha}\\
\frac{\partial g(\cdot)}{\partial K_t} &= \frac{a(1-\alpha)A\alpha K_t^{\alpha-1}\left(a\frac{1+\pi(1+p)}{v}\right)^{-\alpha}}{v}\\
\frac{\partial g(\cdot)}{\partial \omega_t} &= 0
\end{aligned}$$

Hence, denoting $a\frac{1+\pi(1+p)}{v} \equiv \ell$ we have:

$$J = \begin{pmatrix} \begin{bmatrix} N\left[\pi(1+p)\left(\frac{a(1-\alpha)A\alpha K_t^{\alpha-1}(\ell)^{-\alpha}}{v}\right)\right] \\ +N\pi\left[p\alpha A(\alpha-1)K_t^{\alpha-2}(\ell)^{1-\alpha}\omega_t\right] \end{bmatrix} & N\pi p\alpha AK_t^{\alpha-1}(\ell)^{1-\alpha} \\ \frac{a(1-\alpha)A\alpha K_t^{\alpha-1}(\ell)^{-\alpha}}{v} & 0 \end{pmatrix}$$

The determinant is:

$$\det(J) = -N\pi\frac{a}{v}p(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha}$$

We thus have eigenvalues of opposite signs.

The trace is:

$$tr(J) = N\left[\pi(1+p)\left(\frac{a(1-\alpha)A\alpha K^{\alpha-1}(\ell)^{-\alpha}}{v}\right)\right] + N\pi\left[p\alpha A(\alpha-1)K^{\alpha-2}(\ell)^{1-\alpha}\omega_t\right]$$

Note that, at the steady-state, we have:

$$\begin{aligned} N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK^{\alpha}(\ell)^{-\alpha}}{v}\right)\right] + N\pi\left[p\alpha AK^{\alpha-1}(\ell)^{1-\alpha}\omega_t\right] &= K \\ \alpha N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK^{\alpha-1}(\ell)^{-\alpha}}{v}\right)\right] &= \alpha\left[1-N\pi\left[p\alpha AK^{\alpha-2}(\ell)^{1-\alpha}\omega_t\right]\right] \end{aligned}$$

Hence the trace is:

$$tr(J) = \alpha - N\pi\left[\frac{a(1-\alpha)p\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right]$$

The conditions for stability are (see Medio and Lines 2001):

$$\begin{aligned} \text{(i) } 1+tr(J)+\det(J) &> 0 \\ \text{(ii) } 1-tr(J)+\det(J) &> 0 \\ \text{(iii) } 1-\det(J) &> 0 \end{aligned}$$

In this setting the conditions are:

$$\begin{aligned} \text{(i) } 1+\alpha-N\pi\left[\frac{a(1-\alpha)p\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right] - N\pi\frac{a}{v}p(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha} &> 0 \\ \text{(ii) } 1-\alpha+N\pi\left[\frac{a(1-\alpha)p\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right] - N\pi\frac{a}{v}p(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha} &> 0 \\ \text{(iii) } 1+N\pi\frac{a}{v}p(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha} &> 0 \end{aligned}$$

Condition (iii) is satisfied.

Regarding condition (i), this is satisfied iff:

$$\begin{aligned} 1+\alpha-N\pi\left[\frac{ap(1-\alpha)\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right] &> N\pi\frac{ap(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v} \\ &\iff \\ 1 &> N\pi\left[\frac{ap(1-\alpha)\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right] \end{aligned}$$

Note that, at the steady-state, we have:

$$N\left[\pi(1+p)\left(\frac{a(1-\alpha)AK^{\alpha-1}(\ell)^{-\alpha}}{v}\right)\right]+N\pi\left[p\alpha AK^{\alpha-2}(\ell)^{1-\alpha}\omega_t\right]=1$$

thus

$$N\pi(1+p)\left(\frac{a(1-\alpha)AK^{\alpha-1}(\ell)^{-\alpha}}{v}\right)=1-N\pi\left[\frac{ap(1-\alpha)\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right]$$

The LHS being positive, so is the RHS. Hence condition (i) is satisfied.

Consider now condition (ii).

$$1-\alpha+N\pi\left[\frac{a(1-\alpha)p\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right]-N\pi\frac{a}{v}p(1-\alpha)\alpha^2A^2K^{2\alpha-2}(\ell)^{1-2\alpha}>0$$

This is true iff:

$$(1-\alpha)\left(1+N\pi\left[\frac{a(1-\alpha)p\alpha A^2K^{2\alpha-2}(\ell)^{1-2\alpha}}{v}\right]\right)>0$$

That condition is also satisfied. Thus $K^{s**}>0$ is locally stable.

**Reverse retirement** Under reverse retirement, borrowing and saving satisfy the following conditions:

$$\begin{aligned}\frac{1}{-s_t} &= R_{t+1}\frac{1}{\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}-z_{t+1}}\rightarrow s_t=\frac{\tilde{w}_{t+1}\tilde{\ell}_{t+1}}{-R_{t+1}\left(1+\frac{1}{\pi}\right)}-\frac{z_{t+1}}{-R_{t+1}\left(1+\frac{1}{\pi}\right)}\\ \frac{1}{\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}-z_{t+1}\right)} &= \frac{R_{t+2}}{\frac{R_{t+2}z_{t+1}}{p}}\rightarrow z_{t+1}=\frac{p}{1+p}\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}\right)\end{aligned}$$

Hence

$$s_t=\frac{\tilde{w}_{t+1}\tilde{\ell}_{t+1}}{-R_{t+1}\left(1+\frac{1}{\pi}\right)}+\frac{\frac{p}{1+p}\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}\right)}{R_{t+1}\left(1+\frac{1}{\pi}\right)}=-\tilde{w}_{t+1}\tilde{\ell}_{t+1}\frac{\pi}{1+\pi+p\pi}\frac{1}{R_{t+1}}$$

Hence

$$z_{t+1}=\frac{p}{1+p}\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}-\tilde{w}_{t+1}\tilde{\ell}_{t+1}\frac{1}{1+\pi+p\pi}\right)=\tilde{w}_{t+1}\tilde{\ell}_{t+1}\left(\frac{\pi p}{1+\pi+p\pi}\right)$$

Note that $\tilde{\ell}_{t+1}$ satisfies:

$$\tilde{w}_{t+1}\frac{1}{\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}-z_{t+1}\right)}=\tilde{v}\rightarrow\tilde{\ell}_{t+1}=\frac{1}{\tilde{v}}-\frac{R_{t+1}s_t}{\tilde{w}_{t+1}\pi}+\frac{z_{t+1}}{\tilde{w}_{t+1}}$$

Hence

$$\tilde{\ell}_{t+1}=\frac{1}{\tilde{v}}-\frac{R_{t+1}s_t}{\tilde{w}_{t+1}\pi}+\frac{\frac{p}{1+p}\left(\tilde{w}_{t+1}\tilde{\ell}_{t+1}+\frac{R_{t+1}s_t}{\pi}\right)}{\tilde{w}_{t+1}}=\frac{(1+p)}{\tilde{v}}-\frac{R_{t+1}s_t}{\tilde{w}_{t+1}\pi}$$

Hence, substituting for $\tilde{\ell}_{t+1}$ in savings $s_t$, we obtain:

$$s_t = -\tilde{w}_{t+1}\tilde{\ell}_{t+1}\frac{\pi}{1+\pi+p\pi}\frac{1}{R_{t+1}} = -\frac{\tilde{w}_{t+1}}{R_{t+1}}\frac{1}{\tilde{v}}$$

Hence

$$\tilde{\ell}_{t+1} = \frac{(1+p)}{\tilde{v}} - \frac{R_{t+1}}{\tilde{w}_{t+1}\pi}\left(-\frac{\tilde{w}_{t+1}}{R_{t+1}}\frac{1}{\tilde{v}}\right) = \frac{1+\pi+\pi p}{\tilde{v}\pi}$$

Hence

$$z_{t+1} = \tilde{w}_{t+1}\left(\frac{1+p}{\tilde{v}} + \frac{1}{\tilde{v}\pi}\right)\left(\frac{\pi p}{1+\pi+p\pi}\right) = \tilde{w}_{t+1}\frac{p}{\tilde{v}}$$

Remind that: $\tilde{w}_t = bF_L = b(1-\alpha)AK_t^{\alpha}(\pi b\tilde{\ell}_t)^{-\alpha} = b(1-\alpha)AK_t^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}$ and $R_t = F_K = \alpha AK_t^{\alpha-1}(\pi b\tilde{\ell}_t)^{1-\alpha} = \alpha AK_t^{\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{1-\alpha}$.

Substituting for these in the capital accumulation equation $K_{t+1} = Ns_t + N\pi z_t$ yields:

$$\begin{aligned}
K_{t+1} &= -N\frac{\tilde{w}_{t+1}}{R_{t+1}}\frac{1}{\tilde{v}} + N\pi\tilde{w}_t\frac{p}{\tilde{v}} \\
&= \left[\begin{array}{l} -N\frac{b(1-\alpha)AK_{t+1}^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}}{\alpha AK_{t+1}^{\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{1-\alpha}}\frac{1}{\tilde{v}} \\ +N\pi b(1-\alpha)AK_t^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}} \end{array}\right] \\
&\Longleftrightarrow \\
K_{t+1}\left[1 + N\frac{b(1-\alpha)A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}}{\alpha A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{1-\alpha}}\frac{1}{\tilde{v}}\right] &= N\pi b(1-\alpha)AK_t^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}} \\
K_{t+1} &= \frac{N\pi b(1-\alpha)AK_t^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{1 + N\frac{b(1-\alpha)A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}}{\alpha A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{1-\alpha}}\frac{1}{\tilde{v}}}
\end{aligned}$$

Let us denote $\Psi \equiv 1 + N\frac{b(1-\alpha)A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}}{\alpha A\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{1-\alpha}}\frac{1}{\tilde{v}}$.

We have:

$$K_{t+1} = \frac{N\pi b(1-\alpha)AK_t^{\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} \equiv J(K_t)$$

We see that $J(0) = 0$, so that $K^{r*} = 0$ is a stationary equilibrium.

Moreover, we have:

$$\begin{aligned}
J'(K_t) &= \frac{N\pi b(1-\alpha)A\alpha K_t^{\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} > 0 \\
J''(K_t) &= \frac{N\pi b(1-\alpha)(\alpha-1)A\alpha K_t^{\alpha-2}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} < 0
\end{aligned}$$

We have also:

$$\lim_{K_t \to 0} J'(K_t) = \frac{N\pi b(1-\alpha)A\alpha K_t^{\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} = +\infty$$

Hence the transition function lies above the 45° line in the neighborhood of $K=0$. We obviously have that $K^{r*}=0$ is unstable.

Moreover, we have:

$$\lim_{K_t \to +\infty} \frac{J(K_t)}{K_t} = \frac{N\pi b(1-\alpha)AK_t^{\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} = 0$$

Hence for large levels of the capital stock, the transition function lies below the 45° line.

Hence, by continuity of $J(K_t)$, there exists at least one stationary equilibrium with strictly positive $K^{r**}>0$. This is also unique, since the transition function is concave.

Regarding the stability, the necessary and sufficient condition is:

$$\left|\frac{N\pi b(1-\alpha)A\alpha K^{r**\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi}\right| < 1$$

Note that, at the stationary equilibrium $K^{r**}$, we have:

$$\begin{aligned}
K^{r**} &= \frac{N\pi b(1-\alpha)AK^{r**\alpha}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi} \\
1 &= \frac{N\pi b(1-\alpha)AK^{r**\alpha-1}\left(\pi b\frac{1+\pi+\pi p}{\tilde{v}\pi}\right)^{-\alpha}\frac{p}{\tilde{v}}}{\Psi}
\end{aligned}$$

Hence the stability condition vanishes to $|\alpha|<1$, which is satisfied. Thus the high stationary equilibrium $K^{r**}>0$ is locally stable.

## 9.3 Proof of Proposition 3

The utilitarian social planner's problem can be rewritten by means of the following Lagrangian:

$$\max_{c,d,e,\ell,\tilde{\ell},K} \begin{bmatrix} N\left[u(c) - v\ell + \pi\left[u(d) - \tilde{v}\tilde{\ell}\right] + \pi p u(e)\right] \\ +\lambda\left[F\left(K, aN\ell + b\pi N\tilde{\ell}\right) - Nc - \pi N d - \pi p N e - K\right] \end{bmatrix}$$
$$+\rho\ell_t + \varsigma(1-\ell_t) + \varphi\tilde{\ell}_{t+1} + \psi(1-\tilde{\ell}_{t+1})$$

FOCs are:

$$\begin{aligned}
u'(c) &= \lambda = u'(d) = u'(e) \\
-Nv + \lambda F_L\left(K, aN\ell + b\pi N\tilde{\ell}\right)aN + \rho - \varsigma &= 0 \\
-\pi N\tilde{v} + \lambda F_L\left(K, aN\ell + b\pi N\tilde{\ell}\right)b\pi N + \varphi - \psi &= 0 \\
F_K\left(K, aN\ell + b\pi N\tilde{\ell}\right) &= 1
\end{aligned}$$

as well as conditions

$$\begin{aligned}
\rho &\geq 0, \ell \geq 0 \text{ and } \varsigma \geq 0, 1-\ell \geq 0 \\
\varphi &\geq 0, \tilde{\ell} \geq 0 \text{ and } \psi \geq 0, 1-\tilde{\ell} \geq 0
\end{aligned}$$

with complementary slackness.

Let us compare the two FOCs for $\ell$ and $\tilde{\ell}$:

$$\begin{aligned}
-Nv + \lambda F_L\left(K, aN\ell + b\pi N\tilde{\ell}\right) aN + \rho - \varsigma &= 0 \\
-\pi N\tilde{v} + \lambda F_L\left(K, aN\ell + b\pi N\tilde{\ell}\right) b\pi N + \varphi - \psi &= 0
\end{aligned}$$

Let us suppose that young workers are weakly more productive than old workers.

If $a \geq b$, we have, given $\tilde{v} > v$, that $\tilde{\ell} = 0$ and $\varphi > 0$ and $\psi = 0$. We also have $\rho = 0$ and $\varsigma \geq 0$. Hence standard retirement holds ($\tilde{\ell} = 0$), and we have:

$$\begin{aligned}
u'(c^u) &= u'(d^u) = u'(e^u) \\
u'(c^u)F_L\left(K^u, aN\ell^u\right) a &\geq v \text{ and } F_K\left(K^u, aN\ell^u\right) = 1
\end{aligned}$$

If $a < b$, several cases can arise.

If $\frac{v}{a} < \frac{\tilde{v}}{b}$, we have $\tilde{\ell} = 0$ and $\varphi > 0$ and $\psi = 0$. We also have $\rho = 0$ and $\varsigma \geq 0$. Hence standard retirement holds ($\tilde{\ell} = 0$), and we have:

$$\begin{aligned}
u'(c^u) &= u'(d^u) = u'(e^u) \\
u'(c^u)F_L\left(K^u, aN\ell^u\right) a &\geq v \text{ and } F_K\left(K^u, aN\ell^u\right) = 1
\end{aligned}$$

If $\frac{v}{a} > \frac{\tilde{v}}{b}$, we must have $\ell = 0$ and $\rho > 0$ and $\varsigma = 0$. We also have $\varphi = 0$ and $\psi \geq 0$. Hence reverse retirement holds ($\ell = 0$), and we have:

$$\begin{aligned}
u'(c^u) &= u'(d^u) = u'(e^u) \\
u'(c^u)F_L\left(K^u, b\pi N\tilde{\ell}^u\right) b &\geq \tilde{v} \text{ and } F_K\left(K^u, \pi N\tilde{\ell}^u\right) = 1
\end{aligned}$$

## 9.4 Proof of Proposition 4

Assume a Cobb-Douglas production function and logarithmic utility.

When it involves standard retirement, the utilitarian long-run optimum involves:

$$\begin{aligned}
c &= d = e \rightarrow AK^{\alpha}\left(aN\ell\right)^{1-\alpha} = c(N + \pi N + \pi pN) + K \\
\frac{1}{c} aAK^{\alpha}(1-\alpha)(aN\ell)^{-\alpha} &= v \\
A\alpha K^{\alpha-1}\left(aN\ell\right)^{1-\alpha} &= 1
\end{aligned}$$

Using the fourth condition, we obtain:

$$A\alpha K^{\alpha-1}\left(aN\ell\right)^{1-\alpha} = 1 \rightarrow \ell = \left(\frac{1}{A\alpha K^{\alpha-1}\left(aN\right)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} = \frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

Substituting for this in the third condition, we have:

$$\frac{1}{c}aAK^{\alpha}(1-\alpha)(aN)^{-\alpha}\left(\frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha}=v\rightarrow c^{u}=\frac{aA(1-\alpha)\left(A\alpha\right)^{\frac{\alpha}{1-\alpha}}}{v}$$

Hence, back to the resource constraint, we obtain:

$$\begin{aligned}AK^{\alpha}\left(aN\frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} &= \frac{aA(1-\alpha)N\left(A\alpha\right)^{\frac{\alpha}{1-\alpha}}(1+\pi+\pi p)}{v}+K \\ &\rightarrow \quad K^{u}=\frac{a\alpha AN\left(A\alpha\right)^{\frac{\alpha}{1-\alpha}}(1+\pi+\pi p)}{v}\end{aligned}$$

Hence we obtain:

$$\ell^{u}=\frac{\frac{a\alpha AN(A\alpha)^{\frac{\alpha}{1-\alpha}}(1+\pi+\pi p)}{v}}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}=\frac{1+\pi+\pi p}{v}$$

When the utilitarian optimum involves reverse retirement, we have:

$$\begin{aligned}c &= d=e\rightarrow AK^{\alpha}\left(b\pi N\tilde{\ell}\right)^{1-\alpha}=c(N+\pi N+\pi pN)+K \\ \frac{1}{c}bAK^{\alpha}(1-\alpha)(\pi Nb\ell)^{-\alpha} &= \tilde{v} \\ A\alpha K^{\alpha-1}\left(\pi Nb\tilde{\ell}\right)^{1-\alpha} &= 1\end{aligned}$$

Using the fourth condition, we obtain:

$$A\alpha K^{\alpha-1}\left(\pi Nb\tilde{\ell}\right)^{1-\alpha}=1\rightarrow\tilde{\ell}=\frac{K}{\pi Nb}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

Substituting for this in the third condition, we have:

$$c^{u}=\frac{bA(1-\alpha)\left(A\alpha\right)^{\frac{\alpha}{1-\alpha}}}{\tilde{v}}$$

Hence, back to the resource constraint, we obtain:

$$\begin{aligned}AK^{\alpha}\left(b\pi N\frac{K}{\pi Nb}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} &= \frac{bA(1-\alpha)\left(A\alpha\right)^{\frac{\alpha}{1-\alpha}}}{\tilde{v}}(N+\pi N+\pi pN)+K \\ &\rightarrow \quad K^{u}=\frac{b\left(A\alpha\right)^{\frac{1}{1-\alpha}}}{\tilde{v}}(N+\pi N+\pi pN)\end{aligned}$$

Hence

$$\tilde{\ell}^{u}=\frac{1+\pi+\pi p}{\tilde{v}\pi}$$

This completes the proof of Proposition 4.

## 9.5 Proof of Proposition 5

The *ex post* egalitarian social planning problem can be rewritten by means of the following Lagrangian:

$$\begin{aligned}&\max_{c,d,e,\ell,\tilde{\ell},K} N\left[u\left(c\right)-v\ell\right]\\&+\lambda\left[F\left(K,aN\ell+b\pi N\tilde{\ell}\right)-Nc-\pi Nd-p\pi Ne-K\right]\\&+\mu\left[u(d)-\tilde{v}\tilde{\ell}\right]+\sigma\left[u(e)\right]+\rho\ell+\varsigma(1-\ell)+\varphi\tilde{\ell}+\psi(1-\tilde{\ell})\end{aligned}$$

FOCs yield:

$$\begin{aligned}Nu'(c) &= \lambda N \text{ and } \lambda\pi N=\mu u'(d) \text{ and } \lambda p\pi N=\sigma u'(e)\\ Nv &= \lambda F_L\left(K,aN\ell+b\pi N\tilde{\ell}\right)aN+\rho-\zeta\\ \mu\tilde{v} &= \lambda F_L\left(K,aN\ell+b\pi N\tilde{\ell}\right)b\pi N+\varphi-\psi\\ F_K\left(K,aN\ell+b\pi N\tilde{\ell}\right) &= 1\end{aligned}$$

as well as conditions

$$\begin{aligned}\mu &\geq 0, u(d)-\tilde{v}\tilde{\ell}\geq 0 \text{ and } \sigma\geq 0, u(e)\geq 0\\ \rho &\geq 0, \ell\geq 0 \text{ and } \varsigma\geq 0, 1-\ell\geq 0\\ \varphi &\geq 0, \tilde{\ell}\geq 0 \text{ and } \psi\geq 0, 1-\tilde{\ell}\geq 0\end{aligned}$$

with complementary slackness.

Note that: $\mu=\frac{\pi Nu'(c)}{u'(d)}$ is the shadow value of relaxing the old-age egalitarian constraint.

Let us compare the two FOCs for $\ell$ and $\tilde{\ell}$:

$$\begin{aligned}\lambda F_L\left(K,aN\ell+b\pi N\tilde{\ell}\right)N &= \frac{Nv}{a}-\frac{\rho}{a}+\frac{\zeta}{a}\\ \lambda F_L\left(K,aN\ell+b\pi N\tilde{\ell}\right)N &= \frac{\mu\tilde{v}}{\pi b}-\frac{\varphi}{\pi b}+\frac{\psi}{\pi b}\end{aligned}$$

If $\frac{v}{a}<\frac{\mu\tilde{v}}{\pi Nb}$, we have $\tilde{\ell}=0$ as well as $\varphi>0$ and $\psi=0$. We also have $\rho=0$ and $\varsigma\geq 0$. Hence standard retirement holds ($\tilde{\ell}=0$), and we have:

$$\begin{aligned}c^e &> d^e=\bar{c}=e^e\\ u'(c^e)F_L\left(K^e,aN\ell^e\right) &\geq \frac{v}{a} \text{ and } F_K\left(K^e,aN\ell^e\right)=1\end{aligned}$$

If $\frac{v}{a}>\frac{\mu\tilde{v}}{\pi Nb}$, we have $\ell=0$ as well as $\varphi=0$ and $\psi\geq 0$. We also have $\rho>0$ and $\varsigma=0$. Hence reverse retirement holds ($\ell=0$), and we have:

$$\begin{aligned}c^e &> d^e=u^{-1}(\tilde{v}\tilde{\ell}^e)>e^e=\bar{c}\\ u'(c^e)F_L\left(K^e,b\pi N\tilde{\ell}^e\right) &\geq \frac{\mu\tilde{v}}{\pi Nb} \text{ and } F_K\left(K^e,b\pi N\tilde{\ell}^e\right)=1\end{aligned}$$

## 9.6 Proof of Proposition 6

Assume a Cobb-Douglas production function and logarithmic utility. From egalitarian constraints, we have:

$$d = u^{-1}(\tilde{v}\tilde{\ell}) = \exp(\tilde{v}\tilde{\ell} + \beta) \text{ and } e = u^{-1}(0) = \exp(\beta)$$

Assuming $\max\left\{\ell, \tilde{\ell}\right\} < 1$, the social planning problem is:

$$\max_{c,\ell,\tilde{\ell},K} N\left[\log(c) - \beta - v\ell\right] + \lambda\left[AK^{\alpha}(aN\ell + b\pi N\tilde{\ell})^{1-\alpha} - Nc - \pi N\exp(\tilde{v}\tilde{\ell} + \beta) - p\pi N\exp(\beta) - K\right]$$

FOCs are:

$$\begin{aligned}
\frac{N}{c} &= N\lambda \\
Nv &= \lambda AK^{\alpha}(1-\alpha)(aN\ell + b\pi N\tilde{\ell})^{-\alpha}aN \\
\lambda\pi N\exp(\tilde{v}\tilde{\ell} + \beta)\tilde{v} &= \lambda AK^{\alpha}(1-\alpha)(aN\ell + b\pi N\tilde{\ell})^{-\alpha}b\pi N \\
A\alpha K^{\alpha-1}(aN\ell + b\pi N\tilde{\ell})^{1-\alpha} &= 1
\end{aligned}$$

We know that two cases can arise. In order to identify the conditions under which each case arises, we will solve the problem backwards, and compute the utility of the short-lived under the two cases, and, then, compare their levels, which will allow us to identify conditions under which those cases arise.

Consider first the case where standard retirement holds. We have:

$$\begin{aligned}
d &= e = \exp(\beta) = \bar{c} \\
c &= \frac{AK^{\alpha}(aN\ell)^{1-\alpha} - \pi N(1+p)\bar{c} - K}{N} \\
cv &= AK^{\alpha}(1-\alpha)(aN\ell)^{-\alpha}a \\
A\alpha K^{\alpha-1}(aN\ell)^{1-\alpha} &= 1
\end{aligned}$$

Using the fourth condition, we obtain:

$$A\alpha K^{\alpha-1}(aN\ell)^{1-\alpha} = 1 \rightarrow \ell = \left(\frac{1}{A\alpha K^{\alpha-1}(aN)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} = \frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

Using the third condition, we have:

$$aAK^{\alpha}(1-\alpha)(aN)^{-\alpha}\left(\frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha} = vc \rightarrow c^{e} = \frac{aA(1-\alpha)(A\alpha)^{\frac{\alpha}{1-\alpha}}}{v}$$

Back to the resource constraint, we obtain:

$$\begin{aligned}
AK^{\alpha}\left(aN\frac{K}{aN}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} &= N\frac{aA(1-\alpha)(A\alpha)^{\frac{\alpha}{1-\alpha}}}{v} + \pi N(1+p)\bar{c} + K \\
&\rightarrow K^{e} = \frac{\alpha}{1-\alpha}N\left(\frac{aA(1-\alpha)(A\alpha)^{\frac{\alpha}{1-\alpha}}}{v} + \pi(1+p)\bar{c}\right)
\end{aligned}$$

The well-being of the short-lived is here:

$$U^{SL} = \log\left(\frac{aA(1-\alpha)\,(A\alpha)^{\frac{\alpha}{1-\alpha}}}{v}\right) - \beta - \left[\frac{\alpha}{1-\alpha}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\left(A(1-\alpha)\,(A\alpha)^{\frac{\alpha}{1-\alpha}} + \frac{v}{a}\pi(1+p)\bar{c}\right)\right]$$

Denoting $\Xi \equiv A(1-\alpha)\,(A\alpha)^{\frac{\alpha}{1-\alpha}} \rightarrow (1-\alpha) = \frac{\Xi}{A(A\alpha)^{\frac{\alpha}{1-\alpha}}}$, we have:

$$U^{sSL} = \log\left(\frac{a}{v}\Xi\right) - \beta - \left[\frac{\alpha\left(\Xi + \frac{v}{a}\pi(1+p)\bar{c}\right)}{(1-\alpha)\,(A\alpha)^{\frac{1}{1-\alpha}}}\right] = \log\left(\frac{a}{v}\Xi\right) - \beta - 1 - \frac{v}{a\Xi}\pi(1+p)\bar{c}$$

Consider now the case where reverse retirement holds. We have:

$$\begin{aligned}
c &> d = u^{-1}(\tilde{v}\tilde{\ell}) = \exp\left(\tilde{v}\tilde{\ell} + \beta\right) > e = \bar{c} = \exp(\beta) \\
AK^{\alpha}\,(aN\ell)^{1-\alpha} &= cN + \pi N \exp\left(\tilde{v}\tilde{\ell} + \beta\right) + \pi N p\bar{c} + K \\
AK^{\alpha}\,(1-\alpha)\,(b\pi N\tilde{\ell})^{-\alpha} b &= \exp(\tilde{v}\tilde{\ell} + \beta)\tilde{v} \\
A\alpha K^{\alpha-1}\left(b\pi N\tilde{\ell}\right)^{1-\alpha} &= 1
\end{aligned}$$

Using the fourth condition, we obtain:

$$A\alpha K^{\alpha-1}\left(b\pi N\tilde{\ell}\right)^{1-\alpha} = 1 \rightarrow \tilde{\ell} = \left(\frac{1}{A\alpha K^{\alpha-1}\,(b\pi N)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} = \frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

Substituting for this in the third condition, we have:

$$\begin{aligned}
AK^{\alpha}\,(1-\alpha)\left(b\pi N\frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{-\alpha} b &= \exp\left(\tilde{v}\frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}} + \beta\right)\tilde{v} \\
&\rightarrow K = \frac{b\pi N\,(A\alpha)^{\frac{1}{1-\alpha}}}{\tilde{v}}\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta\right]
\end{aligned}$$

Hence, back to the resource constraint:

$$\begin{aligned}
AK^{\alpha}\left(b\pi N\tilde{\ell}\right)^{1-\alpha} &= cN + \pi N\exp\left(\tilde{v}\tilde{\ell} + \beta\right) + \pi N p\bar{c} + K \\
AK^{\alpha}\left(b\pi N\frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}}\right)^{1-\alpha} &= \left[\begin{array}{l} cN + \pi N\exp\left(\tilde{v}\frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}} + \beta\right) + \pi N p\bar{c} \\ + \frac{b\pi N(A\alpha)^{\frac{1}{1-\alpha}}\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta\right]}{\tilde{v}} \end{array}\right] \\
&\rightarrow c = \left[\begin{array}{l} \frac{b\pi\left(\frac{1-\alpha}{\alpha}\right)(A\alpha)^{\frac{1}{1-\alpha}}\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta\right]}{\tilde{v}} \\ -\pi\exp\left(\tilde{v}\frac{K}{b\pi N}\left(\frac{1}{A\alpha}\right)^{\frac{1}{1-\alpha}} + \beta\right) - \pi p\bar{c} \end{array}\right]
\end{aligned}$$

since $\Xi \equiv A(1-\alpha)\,(A\alpha)^{\frac{\alpha}{1-\alpha}} \rightarrow (1-\alpha) = \frac{\Xi}{A(A\alpha)^{\frac{\alpha}{1-\alpha}}}$ and $K^{e} = \frac{b\pi N(A\alpha)^{\frac{1}{1-\alpha}}}{\tilde{v}}\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta\right]$, this is equal to:

$$c^{e} = \frac{b\pi\Xi\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta\right]}{\tilde{v}} - \pi\frac{b}{\tilde{v}}\Xi - \pi p\bar{c}$$

Hence the well-being of the short-lived is:

$$U^{rSL} = \log\left(\frac{b\pi\Xi}{\tilde{v}}\left[\log\left(\frac{b}{\tilde{v}}\Xi\right) - \beta - 1\right] - \pi p\bar{c}\right) - \beta$$

Then, whether standard retirement or reverse retirement prevails depends on whether:

$$\max\left\{U^{sSL}, U^{rSL}\right\} = U^{sSL} \text{ or } U^{rSL}$$

This condition corresponds to the one stated in Proposition 6. This completes the proof of that proposition.

## 9.7 Proof of Proposition 7

Suppose the government implements a system of intergenerational lumpsum transfers leading to $K = K^u$ such that $K^u$ satisfies:

$$F_K\left(K^u, aN\ell^u\right) = 1 \text{ and } F_L\left(K^u, aN\ell^u\right) u'\left(c^u\right) = v$$

In the competitive economy, $\ell$ satisfies:

$$wu'\left(c\right) = v = F_L\left(K, aN\ell\right) u'\left(c\right)$$

Hence, if the capital stock is optimal, we have that $K$ satisfies the condition:

$$F_L\left(K, aN\ell\right) u'\left(c\right) = v$$

as well as the condition:

$$F_L\left(K, aN\ell^u\right) u'\left(c^u\right) = v$$

The two conditions are satisfied when $\ell = \ell^u$ and $c = c^u$.[22] As a consequence, we have that, under $K = K^u$ and $\ell = \ell^u$:

$$R = F_K\left(K, aN\ell\right) = F_K\left(K^u, aN\ell^u\right) = 1$$

Hence, from individual's FOCs,

$$u'(c^u) = Ru'(d) = R^2u'(e)$$

we obtain that, given $R = 1$, that:

$$c^u = d = e$$

[22] Indeed, if we had $\ell > \ell^u$, this would imply $F_L\left(K, aN\ell\right) < F_L\left(K, aN\ell^u\right)$, so that the equality of the two conditions would require $u'(c) > u'(c^u)$, that is, $c < c^u$, which would contradict the resource constraint: for a given capital stock, one cannot have a longer working period and less consumption.

## 9.8 Proof of Proposition 8

The prohibition of young-age labor and the imposition of old-age retirement suffice to obtain the optimal levels $\ell = \ell^e = 0$ and $\tilde{\ell} = \tilde{\ell}^e$. One needs also a system of intergenerational lump-sum transfers aimed at achieving the Golden Rule capital level:

$$F_K(K^e, \pi N b \tilde{\ell}^e) = 1$$

Note, however, that those instruments do not suffice here to decentralize the social optimum, since these do not automatically lead to the optimal consumption profile. Clearly, given the prohibition of young-age labor, the young have to borrow some resources to be able to consume. But nothing *a priori* guarantees that they borrow resources that lead to the optimal consumption profile.

To see this, note that the FOC for young-age savings/borrowing satisfies, under the subsidy on borrowing $\theta$:

$$(1+\theta)u'(c) = Ru'(d)$$

whereas the optimal consumption profile satisfies:

$$u'(c^e) = \frac{\mu}{\pi N} u'(d^e)$$

Hence, given that $R = 1$ under the intergenerational transfer device, we have that the subsidy $\theta$ should satisfy:

$$\frac{1}{1+\theta} = \frac{\mu}{\pi N} \to \theta = \frac{\pi N}{\mu} - 1 = \frac{\pi N}{\frac{u'(c^e)\pi N}{u'(d^e)}} - 1 = \frac{u'(d^e)}{u'(c^e)} - 1$$

Let us now consider consumption in periods 3 and 4 of life. The FOC for old-age savings satisfies, under the tax $\tau$:

$$u'(d) = R(1-\tau)u'(e)$$

whereas the optimal profile satisfies:

$$u'(d^e) = \frac{\sigma}{\mu p} u'(e^e)$$

Hence, given that $R = 1$ under the intergenerational transfer device, we have that the tax should satisfy:

$$1-\tau = \frac{\sigma}{\mu p} \to \tau = 1 - \frac{\sigma}{\mu p} = 1 - \frac{\frac{u'(c^e)p\pi N}{u'(e^e)}}{\frac{u'(c^e)\pi N}{u'(d^e)}p} = 1 - \frac{u'(d^e)}{u'(e^e)}$$

Finally, we need lumpsum transfers $T$ and $\tilde{T}$ that will lead to the satisfaction of egalitarian constraints, at the old age and at the very old age:

$$d^{LF} + T = d^e \text{ and } e^{LF} + \tilde{T} = e^e$$

Together with the retirement standards $\ell = \ell^e = 0$ and $\tilde{\ell} = \tilde{\ell}^e$, the intergenerational lumpsum transfers leading to the Golden Rule, as well as with the two taxes on savings (at young age and old age), those lumpsum transfers allow for the decentralization of the *ex post* egalitarian optimum. This completes the proof of Proposition 8.